1 Poincaré duality

If \( M \) is a compact manifold, oriented of dimension \( m \), we can integrate an \( m \)-form in \( M \), and by the stokes' theorem, if the integral of an exact form is equal to zero, then the linear map \( \omega \in \Omega^m(M) \mapsto \int_M \omega \in \mathbb{R} \) induces a linear map

\[
D_M : H^k(M) \to (H^{m-k})^*, \quad D_M[\omega] : [\eta] \mapsto \int_M \omega \wedge \eta
\]

The Poincaré's duality theorem shows that this map is an isomorphism. The proof we will use proves a more general result, for orientable manifolds which are not compact.

If \( \omega \in \Omega^k(M) \) and \( \eta \in \Omega^{m-k}(M) \) are two chains, then \( \omega \wedge \eta \) has compact support and, by the stokes theorem, it follows that

\[
\int_M (\omega + d\theta) \wedge (\eta \wedge d\rho) = \int_M \omega \wedge \eta
\]

and thus, one could define a linear map between \( H^k(M) \) and the dual \((H^{m-k}(M))^*\) of \( H^{m-k}(M)\),

\[
D_M : H^k(M) \to (H^{m-k}(M))^*
\]

**Lemma 1.1 (five lemma)** Let \( f_j : M_j \to M_{j+1}, f'_j : M'_j \to M'_{j+1}, \phi_j : M_j \to M'_j \) be homomorphisms between modulos such that the following diagram commutes and the two horizontal sequences are exact.

\[
\begin{array}{ccccccc}
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\
\phi_1 & \downarrow & \phi_2 & \downarrow & \phi_3 & \downarrow & \phi_4 & \downarrow & \phi_5 \\
M'_1 & \xrightarrow{f'_1} & M'_2 & \xrightarrow{f'_2} & M'_3 & \xrightarrow{f'_3} & M'_4 & \xrightarrow{f'_4} & M'_5 \\
\end{array}
\]

If \( \phi_1, \phi_2, \phi_4, \phi_5 \) are isomorphisms then \( \phi_3 \) is also an isomorphism.

**Proof.** We will show that \( \phi_3 \) is surjective. Let \( y_3 \in M'_3 \), as \( \phi_4 \) is an isomorphism, there exists \( x_4 \in M_4 \) such that \( f'_4(y_3) = \phi_4(x_4) \), by the commutative of the diagram, we have \( \phi_5 f_4(x_4) = f'_4 \phi_4(x_4) \).

As the bottom sequence is exact, we have that \( f'_4 \phi_4(x_4) = f'_4 f'_3(y_3) = 0 \). Thus \( \phi_5 f_4(x_4) = 0 \) and therefore \( f_4(x_4) = 0 \) since \( \phi_5 \) is isomorphism. As the upper sequence is exact, there exists some \( x_3 \) such that \( f_3(x_3) = x_4 \). Then by the commutative of the diagram for another time, we have that \( f'_3 \phi_3(x_3) = \phi_4(x_4) = f'_4(y_3) \). Then we have \( f'_3(\phi_3(x_3) - y_3) = 0 \) and, as the bottom sequence is exact, it follows that there exists \( y_2 \in M'_2 \) such that \( f'_2(y_2) = \phi_3(x_3) - y_3 \).

As \( \phi_2 \) is surjective, there is \( x_2 \in M_2 \) such that \( \phi_2(x_2) = y_2 \). Then \( \phi_4 f_2(x_2) = f'_4 \phi_2(x_2) = \phi_3(x_3) - y_3 \). Therefore, \( x_3 - f_2(x_2) \in M_3 \) and \( \phi_3(x_3 - f_2(x_2)) = y_3 \), which proves that \( \phi_3 \) is surjective. In the same manner we have prove that \( \phi_3 \) is injective.

Let \( M = U \cup V \) where \( U \) and \( V \) are open sets. A form \( \omega \) with support in \( U \) or in \( V \) or in \( U \cap V \) can extends to a form \( \omega^M \) with the same support. Note also that the extension also preserve the property of having compact support. It then follows that the following maps

\[
\alpha : \Omega^k_c(U \cap V) \to \Omega^k_c(U) \oplus \Omega^k_c(V), \omega \mapsto (\omega^M |_U, \omega^M |_V),
\]
β : Ω^k_c(U) ⊕ Ω^k_c(V) → Ω^k_c(M), (ω_1, ω_2) → ω_1^M - ω_2^M

lead to the short exact sequence:

0 → Ω^k_c(U ∩ V) → Ω^k_c(U) ⊕ Ω^k_c(V) → Ω^k_c(M) → 0

this sequence induces the long exact sequence of cohomology group, called the Meyer-Vietoris sequence of cohomology of compact support,

⋯ → H^k_c(U ∩ V) → H^k_c(U) ⊕ H^k_c(V) → H^k_c(M) → H^{k+1}_c(U ∩ V) → ⋯

and, taking their dual, it follows an exact sequence:

⋯ → (H^k_c(U ∩ V))^* → (H^k_c(U) ⊕ H^k_c(V))^* → (H^k_c(M))^* → (H^{k+1}_c(U ∩ V))^* → ⋯

Lemma 1.2 The following diagram is commutative and the vertical sequences are exact.

\[
\begin{array}{ccc}
H^{r-1}(U) \oplus H^{r-1}(V) & \xrightarrow{D_U \oplus D_V} & H^{m-r+1}(U)^* \oplus H^{m-r+1}(V)^* \\
\beta \downarrow & & \downarrow \alpha^* \\
H^{r-1}(U \cap V) & \xrightarrow{D_{U \cap V}} & H^{m-r+1}(U \cap V)^* \\
\alpha \downarrow & & \downarrow \delta^* \\
H^r(M) & \xrightarrow{D_M} & H^{m-r}(M)^* \\
\beta \downarrow & & \downarrow \beta^* \\
H^r(U \oplus H^r(V) & \xrightarrow{D_U \oplus D_V} & H^{m-r}(U \cap V)^* \\
\end{array}
\]

Proof. The proof is trivial, thought one needs to notice that the “−” appears in the first horizontal map because of the definition of β, which sends (ω_1, ω_2) to ω|_{U \cap V} - ω|_{U \cap V}. The (−1)^{r−1} in the second vertical map is due to the identity dω ∧ η = (−1)^{r−1}ω ∧ dη if ω is a r-form.

Lemma 1.3 If B is a base of open sets of M such that if U, V ∈ B then U ∩ V ∈ B and D_U is an isomorphism for each U ∈ B then D_M is an isomorphism.

Proof. Let F be the family of finite unions of the elements from the base B. By the previous lemma and the five lemma, it follows that D_W is an isomorphism if W is a union of two elements U_1 and U_2 of B by hypothesis U_1 ∩ U_2 also belongs to B. By induction, we conclude that D_W is an isomorphism for all elements of F.

We claim that if M = ∪_{i=1}^∞ M_i where these M_i’s are pairwise disjoint open subsets, and each D_{M_i} is an isomorphism, then \( D_M \) is an isomorphism. In fact, as the sets are pairwise disjoint, it follows that \( H^r(M) = \prod_i H^r(M_i) \) and \( H^r_c(M) = \oplus_i H^r_c(M) \). Thus,

\[
(H^r_c(M))^* = \prod_i H^r_c(M)^* 
\]

and

\[
D_M([\omega_i]) = (D_M([\omega_i]))_i 
\]

which proves the statement. To conclude the proof of the lemma, it suffices to write M as a countable union of open sets V_i such that each V_i belongs to F and \( V_i \cap V_{i+j} = \emptyset \) if \( j \geq 2 \) and take \( U = \cup_i V_{2i} \) and \( V = \cup_i V_{2i+1} \). This can be done by standard procedure of defining M = \( \cup_i K_i \) as the countable union of compact sets, one being contained in the interior of another. Then take V_i as the finite cover of the compact set \( K_i \setminus \text{Int}(K_{i-1}) \) using elements of the base B which is also required to be contained in \( \text{Int}(K_{i+1}) \setminus K_{i-2} \).
Lemma 1.4 If $M$ is diffeomorphic to $\mathbb{R}^m$ then $D_M$ is an isomorphism.

Proof. If $0 < r \leq m$ then $H^r(M) = 0$ and $H^{m-r}(M) = 0$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a $C^\infty$ function with compact support whose integral equals 1. Then the constant function 1 can be regarded as a representative of an element in $H^0(M)$, while $f(x)dx$ is a representative of an element of $H^m_c(M)$. As $D_M(1)(f(x)dx^1 \wedge \cdots \wedge dx^m) = \int f(x_1, \cdots, x_n)dx_1 \cdots dx_n = 1$, it follows that 1 is a generator of $H^0(M)$ and the form $f dx$ is a generator of $H^m_c(M)$, thus $D_M$ is an isomorphism.

Lemma 1.5 If $M \subset \mathbb{R}^m$ is an open subset, then $D_M$ is an isomorphism.

Proof. Let $B$ be the collection of the rectangles $I_1 \times \cdots \times I_m \subset M$ where $I_j \subset \mathbb{R}$ are integers. Then the intersection of two elements of $B$ belongs to $B$ and $D_B$ is isomorphism because all the $B \in B$ is diffeomorphic to $\mathbb{R}^m$.

The Poincaré’s duality theorem follows from the previous lemmas.

2 The de Rham theorem

We will prove the de Rham theorem, which establishes the isomorphism between the de Rham cohomology group and the singular cohomology of a manifold. The proof is similar with that of the Poincaré duality theorem.

Consider the subcomplex $C^{\infty}_r$ of the chain complex $C_r(M)$ consisting of all the chains $c = \sum_i a_i \sigma_i$ where each $\sigma + i : \Delta_k \to M$ is of class $C^\infty$ in the sense that it has a $C^\infty$ extension to a neighborhood of $\Delta_k$ in $\mathbb{R}^{k+1}$. Since if $c \in C^{\infty}_k(M)$, then $\partial c \in C^{\infty}_{k-1}(M)$ it follows that the corresponding homology group is $H^{\infty}_k(M)$. Using the barycentric subdivision and the prism operator of the previous notes, one can prove the following:

Lemma 2.1 The inclusion $C^{\infty}_k(M) \hookrightarrow C_k(M)$ induces isomorphisms of the homology groups.

If $c = \sum_i a_i \sigma_i \in C^{\infty}_k(M)$ and $\omega \in \Omega_k(M)$, then define

$$\int_c \omega = \sum_i a_i \int_{\Delta_k} \sigma_i^* \omega.$$  

Observe that as the simplex $\Delta_k$ is orientable, not necessarily consistent with the orientation of $M$. We will show the following version of the stokes theorem which applies to chains in manifolds, regardless of the orientability of compactness.

Theorem 2.2 (the stokes theorem on chains)

$$\int_{\partial c} \omega = \int_c d\omega$$

Proof. by linearity, it suffices to prove

$$\int_{\Delta_k} d\omega = \int_{\partial \Delta_k} \omega$$

Take one point $x_0$ in the interior of the simplex $\Delta_k$ and let $S$ be a sphere with center $x_0$ in the affine subspace $E$ of dimension $k$ which contains the simplex $\Delta_k$. The ray from the origin $x_0$ passing
through one point \( x \in S \) meets the boundary of a simplex in one unique point \( f_0(x) \). Then the function \( f_0 \) is a homeomorphism of \( S \) and the boundary of \( \Delta_k \). Let \( \rho : S \to \mathbb{R} \) be a positive function such that \( f_0(x) = x_0 + \rho(x)(x-x_0) \). If \( \Delta_i \) is the \( i \)-th face of the simplex \( \Delta_k \) and \( S_i = f_0^{-1}(\Delta_i) \) then the restriction of \( \rho \) on \( S_i \) extends to a \( C^\infty \) map \( \rho_i \), from a neighborhood of \( S_i \) in \( S \), \( x \to x_0 + \rho_i(x)(x-x_0) \) belongs to the affine subspace which contains \( \Delta_i \).

We now claim that there exists a constant \( C > 0 \), such that for any \( \delta > 0 \) there exists a function \( \phi_i^\delta : S \to [0, 1] \) of class \( C^\infty \) such that

1. \( \phi_i^\delta(x) = 1 \) if \( x \in S_i \);
2. \( \phi_i(x) = 0 \) if the distance of \( x \) and \( S_i \) is greater than \( 10\sqrt{k+1}\delta \);
3. The norm of the derivative of \( \phi_i^\delta \) at any point is smaller than or equal to \( C/\delta \).

We now proceed to show how this claim can help us to prove the theorem. Consider the \( C^\infty \) map

\[
\rho_S = \sum_i \psi_i^\delta(x) \rho_i(x)
\]

where

\[
\psi_i^\delta(x) = \frac{\phi_i^\delta(x)}{\sum_j \phi_j^\delta(x)}
\]

By the chain rule, there exists a constant \( C' \), independent with \( \delta \) such that the norm of the derivative of each function \( \phi_i^\delta \) is bounded by \( C'/\delta \).

There exists a constant \( C'' > 0 \), independent of \( \delta \), such that the norm of the derivative of \( \rho_S \) at each point is bounded by \( C'' \). In fact, in a neighborhood of one point of \( S_i \) we can write

\[
\rho_S(x) = \rho_i(x) + \sum_{j \neq i} \psi_j^\delta(x)(\rho_j(x) - \rho_i(x))
\]

If \( D\psi_j^\delta(x) \neq 0 \) then the distance of \( x \) to \( S_j \) is smaller than \( 5\sqrt{k+1}\delta \), and therefore, \( |\rho_j(x) - \rho_i(x)| \) is smaller than a constant times \( \delta \) because \( \rho_j - \rho_i \) is Lipschitz and vanishes in \( S_i \cap S_j \). So the derivative of \( \rho_S \) at point \( x \) is bounded by a constant independent of \( \delta \).

Let \( W_\delta \) be the manifold with boundary consisting of points of the form \( x_0 + t(x-x_0) \) with \( x \in S \) and \( 0 \leq t \leq \rho_0(x) \). Let \( S_i(\delta) \) be the subset of the points of \( S_i \) whose distance to each \( S_j, j \neq i \) is greater than \( 5\sqrt{k+1}\delta \). Then the restriction of \( \rho_i \) on \( S_i(\delta) \) coincides with the restriction of \( \rho \) on \( f_0(S_i(\delta)) \subset \Delta_i \). Let \( f_\delta(x) = x_0 + \rho_\delta(x)(x-x_0) \), then \( f_\delta \) is a diffeomorphism between \( S \) and \( \partial W_\delta \) and its restriction on \( S_i(\delta) \) coincides with the restriction of \( f_0 \). Moreover, the derivative of \( f_\delta \) at each point is bounded by a constant independent of \( \delta \). Thus the integral of \( \omega \) in \( f_0(S_i(\delta)) \) is equal to the integral of \( f_\delta^*\omega \) in \( S_i(\delta) \) and, as the derivative of \( f_\delta \) is bounded and the area of \( S_i(\delta) \) tends to zero when \( \delta \to 0 \), then the integral of \( f_\delta^*\omega \) in \( S_i(\delta) \) tends to zero when \( \delta \to 0 \) as is the integral of \( \omega \) in \( \partial W_\delta \cup S_i(\delta) \). Thus the integral of \( \omega \) in the boundary of \( W_\delta \) converges to the integral of \( \omega \) in the boundary of \( \Delta_k \). On the other hand, as \( \rho_\delta \) converges uniformly to \( \rho \) as \( \delta \to 0 \) it follows that the integral of \( d\omega \) converges to the integral of \( \omega \) in \( \Delta_k \), which completes the proof.

We now prove the claim. Let \( \mathbb{Z}^{k+1} \subset \mathbb{R}^{k+1} \) be the set of points with integer coordinates. Then the open balls with centers at points in \( \mathbb{Z}^{k+1} \) at radius \( 2\sqrt{k+1} \) cover \( \mathbb{R}^{k+1} \), moreover there exists a constant \( N_k \) such that for each ball with center in \( \mathbb{Z}^{k+1} \) and radius \( 4\sqrt{k+1} \), the number of points in \( \mathbb{Z}^{k+1} \) that are centers of balls of radius \( 4\sqrt{k+1} \) that intersect the initial ball is smaller or equal to \( N_k \). Take the image of these balls under the linear map \( x \in \mathbb{R}^{k+1} \to \delta x \) it follows that the same properties for the balls \( B_\delta^{j,\lambda} = B(\lambda, 2j\delta\sqrt{k+1}), \lambda \in \delta\mathbb{Z}^{k+1} \) and \( j = 1, 2 \).

Let \( \phi : \mathbb{R}^{k+1} \to [0, 1] \) be a \( C^\infty \) function which vanishes out of the ball with center origin and radius 2 and equals 1 at points in the ball of radius 1. Composing \( \phi \) with the affine diffeomorphism which takes the ball \( B_\lambda^{j,\lambda} \) to the ball with radius 1 and with center in the origin, we obtain a \( C^\infty \) map.
\(\phi_\lambda\) which vanishes out of \(B^{3i}_\lambda\) equals to 1 in \(B^{3i}_\lambda\), and whose derivative at each point is bounded by a constant times the inverse of \(\delta\) and this constant does not depend on \(\delta\) or \(\lambda\). Consider the partition of unity \(\psi_\lambda = \sum \phi_\lambda\). By the chain rule, there exists a constant, independent with \(\delta\) and depends only on the previous constants and \(N_{k+1}\) such that the derivative of \(\psi_\lambda\) is bounded by this constant times the inverse of \(\delta\). For each \(i\) let \(U_i\) be the set of points whose distance to \(S_i\) is smaller than \(10\delta\sqrt{k+1}\). Then define the function \(\phi^i_\lambda(x) = \sum \phi_\lambda(x)\) for each \(\lambda\) such that \(B^{3i}_\lambda \subset U_i\). As all the balls \(B^{3i}_\lambda\) that intersect \(S_i\) is contained in \(U_i\) it follows that \(\phi^i_\lambda(x) = 1\) for \(x \in S_i\) and, as in the neighborhood of each point the number of portions is bounded by \(N_k\) it follows that the derivative of \(\phi^i_\lambda\) ar each point is bounded by the product of a constant with the inverse of \(\delta\), which proves the claim.

Now consider the singular complex cochains, \(\delta : C^q_\infty(M) \to C^{q+1}_\infty(M)\), where \(C^q_\infty(M)\) is the dual of \(C^q(M)\) and \(\delta\) is the dual of the boundary operator \(\partial\), i.e., \(\delta(c^q)(c_r) = c^{q}(\partial c_q)\) for all \(c^q \in C^q_\infty\) and \(c_r \in C^r_\infty\). Let \(d_M : \Omega^q(M) \to C^q_\infty(M)\) be defined by \(d_M(\omega) : c \in \Omega^q(M) \mapsto \int c \omega\). By the stokes theorem it follows that \(d_M \circ \delta = \delta \circ d_M\) and, therefore, define an homomorphism

\[d_M : H^r_{dR}(M) \to H^q_\infty(M)\]

**Lemma 2.3** Let \(M = U \cup V\), where \(U\) and \(V\) are open sets. Consider the following diagram where the two vertical sequences are Meyer-Vietoris sequences and the horizontal homomorphisms are de Rham homomorphism.

\[
\begin{array}{ccc}
H^{r-1}_{dR}(U) \oplus H^r(V) & \xrightarrow{\partial U \oplus \partial V} & H^{m-r+1}_\infty(U) \oplus H^{m-r+1}_\infty(V) \\
\beta \downarrow & & \downarrow \alpha^* \\
H^{r-1}_{dR}(U \cap V) & \xrightarrow{\partial U \cap V} & H^{m-r+1}_\infty(U \cap V) \\
(-1)^r \delta \downarrow & & \downarrow \delta^* \\
H^r_{dR}(M) & \xrightarrow{\partial M} & H^m_\infty(M) \\
\alpha \downarrow & & \downarrow \beta^* \\
H^r_{dR}(U) \oplus H^r(V) & \xrightarrow{\partial U \oplus \partial V} & H^{m-r}_\infty(U \cap V) \\
\beta \downarrow & & \downarrow \alpha^* \\
H^r_{dR}(U \cap V) & \xrightarrow{\partial U \cap V} & H^{m-r}_\infty(U \cap V) \\
\end{array}
\]

Then the diagram commutes.

**Theorem 2.4 (the de Rham theorem)** The homomorphism \(d_M : H^k_{dR}(M) \to H^k_\infty(M)\) defined by integration of the forms in chains is an isomorphism for each manifold \(M\).