Differential Topology notes, 20

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1 The de Rham Cohomology

Propostion 1.1 *If for each* C^∞ *map* $f : S^1 \to M$ *and for each closed* 1*-form* $\omega \in \Omega^1(M)$ *it follows that* $\int_{S^1} f^*\omega = 0$*, then* $H^1(M) = 0$*.*

Proof. Let ω be a close 1-form in M. Suppose $x_0 \in M$, if $x \in M$, take a curve, which is piecewise differentiable, $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$ and $\gamma(1) = x$. Note that if we have two γ and γ' with the same property, then it induces a map $f : S^1 \to M$, then, by hypothesis, we have that $\int_{f(S^1)} \omega = \int_{S^1} f^*\omega = 0$, where $f(S^1)$ is a loop connecting γ and γ'. Thus, we can define $f(x) = \int_\gamma \omega$ in this case. Observe that for any $\alpha : [0,1] \to M$ is a C^∞ curve with $\alpha(0) = x_1$ and $\alpha(1) = x$, holds $f(x) = f(x_1) + \int_\alpha \omega$.

We can take a local chart of x_1 which maps x_1 to 0, such that written by $\widehat{f}$ and $\widehat{\omega}$ as the expression of f and ω, $\widehat{\omega}$ is constant, equal to $\widehat{\omega}(0)$. Then given the curve $t \in [0,1] \mapsto ty$, we have $\widehat{f}(y) - \widehat{f}(0) = \int_\alpha \widehat{\omega} = \int_0^1 \widehat{\omega}(ty)(y)dt = \widehat{\omega}(0)(y)$. So $(D\widehat{f})_0(y) = \widehat{\omega}(0)(y)$. Since x_1 is arbitrarily chosen, it follows that $\omega = df$.

Corollary 1.2 *If* M *is a simply connected manifold then* $H^1(M) = 0$*.*

2 The Mayer-Vietoris sequence

Given a manifold M, $\{U, V\}$ is an open cover by two open sets U, V of M. We show that there exists an exact sequence relating the cohomology groups of M, U, V and $U \cap V$. For each k consider the linear maps:

$$\alpha_k : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V)$$

defined by $\omega \mapsto (\omega|_U, \omega|_V)$, and

$$\beta_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$$

defined by $(\omega_1, \omega_2) \mapsto \omega_1|_{U\cap V} - \omega_2|_{U\cap V}$.

Clearly, α_k is injective and the image of α_k is equal to the kernel of β_k.

Lemma 2.1 *The sequence*

$$0 \to \Omega^k(M) \xrightarrow{\alpha_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta_k} \Omega^k(U \cap V) \to 0$$

is exact.

Proof. We need to show that β_k is surjective. Take a partition of unity λ_U, λ_V subordinate to the cover U, V. If $\omega \in \Omega^k(U \cap V)$ we can define $\omega_1(x) = \lambda_U \omega(x)$ and $\omega_2(x) = -\lambda_v \omega(x)$. Then it is clear that ω_1 and ω_2 are C^∞ forms and $\omega_1|_{U\cap V} - \omega_2|_{U\cap V} = \omega$. ∎

Since the linear transformations α_k and β_k commutes with the boundary operators they induce linear transformations between the cohomology groups which we still denote using the same letters

$$\alpha_k : H^k(M) \to H^k(U) \oplus H^k(V),$$

$$\beta_k : H^k(U) \oplus H^k(V) \to H^k(U \cap V).$$

Theorem 2.2 *There exists a linear map* $\Delta_k : H^k(U \cap V) \to H^{k+1}(M)$ *such that the following Meyer-Vietoris sequence is exact:*

$$\cdots \to H^k(M) \xrightarrow{\alpha_k} H^k(U) \oplus H^k(V) \xrightarrow{\beta_k} H^k(U \cap V) \xrightarrow{\Delta_k} H^{k+1}(M) \to \cdots$$

Proof. Let ω be a closed form in $\Omega^k(U \cap V)$. As β_k is surjective, there exist forms $(\omega_1, \omega_2) \in \Omega^k(U) \oplus \Omega^k(V)$ such that $\omega = \beta_k(\omega_1, \omega_2) = \omega_1|_{U \cap V} - \omega_2|_{U \cap V}$. Since ω is a closed form it follows that for $x \in U \cap V$, $d\omega_1(x) = d\omega_2(x)$. Then, define $\eta(x) = d\omega_1(x)$ if $x \in U$ and $\eta(x) = d\omega_2(x)$ if $x \notin U$. Then $\eta \in \Omega^{k+1}(M)$.

The $k+1$-form η depends on the choice of the forms ω_i. But we will show the follows, actually more general cases. First, its cohomology class does not depend on the choice of different ω_i. Secondly, it does not depend on the choice of the representatives of the cohomology class of ω. $\blacksquare$

A co-chain complex $\mathcal{C}$ is a sequence of vector spaces C^k and linear transformations $d_k : C^k \to C^{k+1}$ such that $d_{k+1} \circ \alpha_k = 0$. A morphism $\alpha : \mathcal{C} \to \mathcal{C}'$ is a family of linear transformations $\alpha_k : C^k \to C^{\prime k}$ such that $d'_k \circ \alpha_k = \alpha_{k+1} \circ d_k$.

A short exact sequence os cochain complexes is a sequence

$$0 \to \mathcal{C} \xrightarrow{\alpha} \mathcal{C}' \xrightarrow{\beta} \mathcal{C}'' \to 0$$

such that the following diagram commute:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C^{k-1} \arrow[r, "\alpha_{k-1}"] \arrow[d, "d_{k-1}"] & C^{\prime k-1} \arrow[r, "\beta_{k-1}"] \arrow[d, "d'_{k-1}"] & C^{\prime\prime k-1} \arrow[r] \arrow[d, "d''_{k-1}"] & 0 \\
0 \arrow[r] & C^{k} \arrow[r, "\alpha_{k}"] \arrow[d, "d_{k}"] & C^{\prime k} \arrow[r, "\beta_{k}"] \arrow[d, "d'_{k}"] & C^{\prime\prime k} \arrow[r] \arrow[d, "d''_{k}"] & 0 \\
0 \arrow[r] & C^{k+1} \arrow[r, "\alpha_{k+1}"] \arrow[d] & C^{\prime k+1} \arrow[r, "\beta_{k+1}"] \arrow[d] & C^{\prime\prime k+1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Now we construct an operator

$$\Delta : H^k(C'') \to H^{k+1}(C)$$

We try to give meaning to the definition as $\Delta = \alpha_{k+1}^{-1} \circ d'_k \circ \beta_k^{-1}$. The process is very similar with the same case in homology, so we omit it. But this method is so important that it gets a name, so called "diagram chasing".

Theorem 2.3 *If* S^n *is a sphere of dimension* $n \geq 1$ *then* $H^k(S^n) = 0$ *if* $k \neq 0, n$ *and* $H^k(S^n) = \mathbb{R}$ *if* $k = 0, n$.

Proof. Let $p, q \in S^1, p \neq q$ and $U = S^1 \backslash \{p\}$ and $V = S^1 \backslash \{q\}$. It follows that $H^1(U) = 0 = H^1(V)$, since both U and V are differmorphic to $\mathbb{R}$ while $H^0(U \cap V) = \mathbb{R}^2$ because $U \cap V$ has two connected components. Thus, the Meyer-Vietoris is as follows,

$$0 \to H^0(S^1) = \mathbb{R} \to H^0(U) \oplus H^0(V) = \mathbb{R}^2 \to H^0(U \cap V) = \mathbb{R}^2 \to H^1(S^1) \to H^1(U) \oplus H^1(V) = 0$$

Compute using the exactness we have that $H^1(S^1) = \mathbb{R}$. Then we denote by Δ the map which for each cohomology class of ω associates the cohomology class of η.

If $n \geq 2, S^n = \mathbb{R}^n \cup \{\infty\}$, for any $p, q \in S^n$, let $U = S^n \backslash \{p\}$ and $V = S^n \backslash \{q\}$, then both U and V are diffeomorphic to $\mathbb{R}^n$. Note that $U \cap V$ is diffeomorphic to $\mathbb{R}^n \backslash \{0\}$. The retraction of $\mathbb{R}^n \backslash \{0\}$ onto S^{n-1} is homotopic to identity, therefore, the cohomology groups of $U \cap V$ and of S^{n-1} are isomorphic. Then we have the following Meyer-Vietoris sequence, for $k \geq 2$,

$$\cdots H^{k-1}(U) \oplus H^{k-1}(V) = 0 \to H^{k-1}(U \cap V) = H^{k-1}(S^{n-1}) \to H^k(S^n) \to H^k(U) \oplus H^k(V) = 0 \cdots$$

which implies that $H^{k-1}(S^{n-1})$ is isomorphic with $H^k(S^n)$. The theorem follows from induction. ∎

If M is an oriented manifold of dimension n, then the integral of an exact n-form is always zero. Thus the linear function $\omega \in \Omega_c^n(M) \mapsto \int_M \omega \in \mathbb{R}$ induces a linear map $I_M : H_c^n(M) \to \mathbb{R}$. This map is obviously surjective because we can take a form whose support is contained in the domain of a local chart and whose expression in this chart is the product of a non negative function and the base $dx^1 \wedge \cdots dx^m$ of $\Lambda^m(\mathbb{R}^m)$ has positive integral.

Corollary 2.4 *An n-form in S^n whose integral vanishes is an exact form.*

Proof. Since $H^n(S^n) = \mathbb{R}^n$ then I_{S^n} is an isomorphism. ∎

Theorem 2.5 $H_c^n(\mathbb{R}^n) = \mathbb{R}$ *and* $H_c^k(\mathbb{R}^n) = 0$ *if* $k < n$.

Proof. Let $1 \leq k < n$. By Poincaré lemma, there exists a $k-1$-form η such that $d\eta = \omega$, we hope to find another $k-1$-form which has compact support.

If $\omega \in \Omega^k(\mathbb{R}^n)$, take an open ball D_R with center at the origin and radius R sufficiently big such that D_R contains the support of ω. Let $A = \mathbb{R}^n \backslash D_{R-\epsilon}$ with ϵ small enough such that A does not intersects the support of ω. The radial projection π of A onto ∂D_R is an homotopic equivalence. It follows that $\pi^* : H^k(S^{n-1}) = H^k(\partial B) \to H^k(A)$ is an isomorphism. On the other hand, we have prove that $H^{k-1}(S^{n-1}) = 0$ and $H^{k-1}(A) = 0$, so each closed $k-1$-form in A is exact. So there exists some $(k-2)$-form λ in A, such that $d\lambda = \eta$ in A.

Let f be a C^∞ function which takes value 0 out of a small neighborhood of $\mathbb{R}^n \backslash D_R$ whose closure is contained in A, and value 1 at a even smaller neighborhood of $\mathbb{R}^n \backslash D_R$. Then define $\widehat{\eta} = d(f\lambda)$ in A and $\widehat{\eta} = 0$ out of A. So $\widehat{\eta}$ is a C^∞ form and $d\widehat{\eta} = 0$ in $\mathbb{R}^n$. $\eta - \widehat{\eta}$ is a form with compact support and $\omega = d(\eta - \widehat{\eta})$, which proves that ω is an exact form.

To show $H_c^n(\mathbb{R}^n) = \mathbb{R}$ it suffices to prove that if $\omega \in \Omega_c^n(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n} \omega = 0$ then $\omega = d\eta$ for some $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$. Take D_R and A as the previous case. We have $\omega = d\eta$ for some $(n-1)$-form η and $d\eta = 0$ in A. On the other hand, by the stokes theorem, it follows that

$$0 = \int_{\mathbb{R}^n} \omega = \int_{D_R} \omega = \int_{D_R} d\eta = \int_{\partial D_R} \eta$$

By the previous corollary, we know that the restriction of η on ∂D_R is an exact form. As the radial projection is an isomorphism between the cohomology groups of A and ∂D_R, we also have that η is an exact form in A. Therefore $\eta = d\lambda$ in A and, therefore, as before, we can find some $\widehat{\eta}$ such that $\omega = d(\eta - \widehat{\eta})$ where $\eta - \widehat{\eta}$ has compact support. So the cohomology class of ω vanishes in $H_c^n(\mathbb{R}^n)$. ∎