1 The de Rham Cohomology

Propostion 1.1 If for each $C^\infty$ map $f : S^1 \to M$ and for each closed 1-form $\omega \in \Omega^1(M)$ it follows that $\int_{S^1} f^* \omega = 0$, then $H^1(M) = 0$.

Proof. Let $\omega$ be a close 1-form in $M$. Suppose $x_0 \in M$, if $x \in M$, take a curve, which is piecewise differentiable, $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$ and $\gamma(1) = x$. Note that if we have two $\gamma$ and $\gamma'$ with the same property, then it induces a map $f : S^1 \to M$, then, by hypothesis, we have that $\int_{f(S^1)} \omega = \int_{S^1} f^* \omega = 0$, where $f(S^1)$ is a loop connecting $\gamma$ and $\gamma'$. Thus, we can define $f(x) = \int_{\gamma} \omega$ in this case. Observe that for any $\alpha : [0,1] \to M$ is a $C^\infty$ curve with $\alpha(0) = x_1$ and $\alpha(1) = x$, holds $f(x) = f(x_1) + \int_{\alpha} \omega$.

We can take a local chart of $x_1$ which maps $x_1$ to 0, such that written by $\hat{f}$ and $\hat{\omega}$ as the expression of $f$ and $\omega$, $\hat{\omega}$ is constant, equal to $\hat{\omega}(0)$. Then given the curve $t \in [0,1] \to ty$, we have $\hat{f}(y) - \hat{f}(0) = \int_{\alpha} \hat{\omega} = \int_0^1 \hat{\omega}(ty)(y)dt = \hat{\omega}(0)(y)$. So $(D\hat{f})_0(y) = \hat{\omega}(0)(y)$. Since $x_1$ is arbitrarily chosen, it follows that $\omega = df$.

Corollary 1.2 If $M$ is a simply connected manifold then $H^1(M) = 0$.

2 The Mayer-Vietoris sequence

Given a manifold $M$, $\{U, V\}$ is an open cover by two open sets $U, V$ of $M$. We show that there exists an exact sequence relating the cohomology groups of $M$, $U, V$ and $U \cap V$. For each $k$ consider the linear maps:

$$\alpha_k : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V)$$

defined by $\omega \mapsto (\omega|_U, \omega|_V)$, and

$$\beta_k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$$

defined by $(\omega_1, \omega_2) \mapsto \omega_1|_{U \cap V} - \omega_2|_{U \cap V}$.

Clearly, $\alpha_k$ is injective and the image of $\alpha_k$ is equal to the kernel of $\beta_k$.

Lemma 2.1 The sequence

$$0 \to \Omega^k(M) \xrightarrow{\alpha_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta_k} \Omega^k(U \cap V) \to 0$$

is exact.

Proof. We need to show that $\beta_k$ is surjective. Take a partition of unity $\lambda_U, \lambda_V$ subordinate to the cover $U, V$. If $\omega \in \Omega^k(U \cap V)$ we can define $\omega_1(x) = \lambda_U(x) \omega(x)$ and $\omega_2(x) = -\lambda_V(x) \omega(x)$. Then it is clear that $\omega_1$ and $\omega_2$ are $C^\infty$ forms and $\omega_1|_{U \cap V} - \omega_2|_{U \cap V} = \omega$. 

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Since the linear transformations $\alpha_k$ and $\beta_k$ commutes with the boundary operators they induce linear transformations between the cohomology groups which we still denote using the same letters

\[ \alpha_k : H^k(M) \to H^k(U) \oplus H^k(V), \]
\[ \beta_k : H^k(U) \oplus H^k(V) \to H^k(U \cap V). \]

**Theorem 2.2** There exists a linear map $\Delta_k : H^k(U \cap V) \to H^{k+1}(M)$ such that the following diagram commute:

\[ \cdots \to H^k(M) \xrightarrow{\alpha_k} H^k(U) \oplus H^k(V) \xrightarrow{\beta_k} H^k(U \cap V) \xrightarrow{\Delta_k} H^{k+1}(M) \to \cdots \]

**Proof.** Let $\omega$ be a closed form in $\Omega^k(U \cap V)$. As $\beta_k$ is surjective, there exist forms $(\omega_1, \omega_2) \in \Omega^k(U) \oplus \Omega^k(V)$ such that $\omega = \beta_k(\omega_1, \omega_2) = \omega_1|_{U \cap V} - \omega_2|_{U \cap V}$. Since $\omega$ is a closed form it follows that for $x \in U \cap V$, $d\omega_1(x) = d\omega_2(x)$. Then, define $\eta(x) = d\omega_1(x)$ if $x \in U$ and $\eta(x) = d\omega_2(x)$ if $x \notin U$. Then $\eta \in \Omega^{k+1}(M)$.

The $k+1$-form $\eta$ depends on the choice of the forms $\omega_i$. But we will show the follows, actually more general cases. First, its cohomology class does not depend on the choice of different $\omega_i$. Secondly, it does not depend on the choice of the representatives of the cohomology class of $\omega$. \hfill \blacksquare

A co-chain complex $\mathcal{C}$ is a sequence of vector spaces $C^k$ and linear transformations $d_k : C^k \to C^{k+1}$ such that $d_{k+1} \circ \alpha_k = 0$. A morphism $\alpha : \mathcal{C} \to \mathcal{C}'$ is a family of linear transformations $\alpha_k : C^k \to C'^k$ such that $d'_k \circ \alpha_k = \alpha_{k+1} \circ d_k$.

A short exact sequence os cochain complexes is a sequence

\[ 0 \to C \xrightarrow{\alpha} C' \xrightarrow{\beta} C'' \to 0 \]

such that the following diagram commute:

\[
\begin{array}{cccccc}
0 & \to & C^{k-1} & \xrightarrow{\alpha_{k-1}} & C'^{k-1} & \xrightarrow{\beta_{k-1}} & C''^{k-1} & \to & 0 \\
\downarrow d_{k-1} & & \downarrow d'_{k-1} & & \downarrow d''_{k-1} & & & & \\
0 & \to & C^k & \xrightarrow{\alpha_k} & C'^k & \xrightarrow{\beta_k} & C''^k & \to & 0 \\
\downarrow d_k & & \downarrow d'_k & & \downarrow d''_k & & & & \\
0 & \to & C^{k+1} & \xrightarrow{\alpha_{k+1}} & C'^{k+1} & \xrightarrow{\beta_{k+1}} & C''^{k+1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\end{array}
\]

Now we construct an operator

\[ \Delta : H^k(C'') \to H^{k+1}(C) \]

We try to give meaning to the definition as $\Delta = \alpha_{k+1} \circ d'_k \circ \beta^{-1}_k$. The process is very similar with the same case in homology, so we omit it. But this method is so important that it gets a name, so called "diagram chasing".

**Theorem 2.3** If $S^n$ is a sphere of dimension $n \geq 1$ then $H^k(S^n) = 0$ if $k \neq 0, n$ and $H^k(S^n) = \mathbb{R}$ if $k = 0, n$.

**Proof.** Let $p, q \in S^1, p \neq q$ and $U = S^1 \setminus \{p\}$ and $V = S^1 \setminus \{q\}$. It follows that $H^1(U) = 0 = H^1(V)$, since both $U$ and $V$ are differomorphic to $\mathbb{R}$ while $H^0(U \cap V) = \mathbb{R}^2$ because $U \cap V$ has two connected components. Thus, the Meyer-Vietoris is as follows,

\[ 0 \to H^0(S^1) = \mathbb{R} \to H^0(U) \oplus H^0(V) = \mathbb{R}^2 \to H^0(U \cap V) = \mathbb{R}^2 \to H^1(U \cap V) \to H^1(U) \oplus H^1(V) = 0 \]
Compute using the exactness we have that $H^1(S^1) = \mathbb{R}$. Then we denote by $\Delta$ the map which for each cohomology class of $\omega$ associates the cohomology class of $\eta$.

If $n \geq 2$, $S^n = \mathbb{R}^n \cup \{\infty\}$, for any $p, q \in S^n$, let $U = S^n \setminus \{p\}$ and $V = S^n \setminus \{q\}$, then both $U$ and $V$ are diffeomorphic to $\mathbb{R}^n$. Note that $U \cap V$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$. The retraction of $\mathbb{R}^n \setminus \{0\}$ onto $S^{n-1}$ is homotopic to identity, therefore, the cohomology groups of $U \cap V$ and of $S^{n-1}$ are isomorphic. Then we have the following Meyer-Vietoris sequence, for $k \geq 2$,

$$\cdots H^{k-1}(U) \oplus H^{k-1}(V) = 0 \rightarrow H^{k-1}(U \cap V) = H^{k-1}(S^{n-1}) \rightarrow H^k(S^n) \rightarrow H^k(U) \oplus H^k(V) = 0 \cdots$$

which implies that $H^{k-1}(S^{n-1})$ is isomorphic with $H^k(S^n)$. The theorem follows from induction. 

If $M$ is an oriented manifold of dimension $n$, then the integral of an exact $n$-form is always zero. Thus the linear function $\omega \in \Omega^n(M) \mapsto \int_M \omega \in \mathbb{R}$ induces a linear map $I_M : H^n(M) \rightarrow \mathbb{R}$. This map is obviously surjective because we can take a form whose support is contained in the domain of a local chart and whose expression in this chart is the product of a non negative function and the base $dx^1 \wedge \cdots dx^n$ of $\Lambda^n(\mathbb{R}^m)$ has positive integral.

**Corollary 2.4** An $n$-form in $S^n$ whose integral vanishes is an exact form.

**Proof.** Since $H^n(S^n) = \mathbb{R}$ then $I_{S^n}$ is an isomorphism. 

**Theorem 2.5** $H^k_c(\mathbb{R}^n) = \mathbb{R}$ and $H^k_c(\mathbb{R}^n) = 0$ if $k < n$.

**Proof.** Let $1 \leq k < n$. By Poincaré lemma, there exists a $k-1$-form $\eta$ such that $d\eta = \omega$, we hope to find another $k-1$-form which has compact support.

If $\omega \in \Omega^k(\mathbb{R}^n)$, take an open ball $D_R$ with center at the origin and radius $R$ sufficiently big such that $D_R$ contains the support of $\omega$. Let $A = \mathbb{R}^n \setminus D_{R-\epsilon}$ with $\epsilon$ small enough such that $A$ does not intersects the support of $\omega$. The radial projection $\pi$ of $A$ onto $\partial D_R$ is an homotopic equivalence. It follows that $\pi^* : H^k(S^{n-1}) = H^k(\partial B) \rightarrow H^k(A)$ is an isomorphism. On the other hand, we have prove that $H^{k-1}(S^{n-1}) = 0$ and $H^{k-1}(A) = 0$, so each closed $k-1$-form in $A$ is exact. So there exists some $(k-2)$-form $\lambda$ in $A$, such that $d\lambda = \eta$ in $A$.

Let $f$ be a $C^\infty$ function which takes value 0 out of a small neighborhood of $\mathbb{R}^n \setminus D_R$ whose closure is contained in $A$, and value 1 at a even smaller neighborhood of $\mathbb{R}^n \setminus D_R$. Then define $\tilde{\eta} = d(f\lambda)$ in $A$ and $\tilde{\eta} = 0$ out of $A$. So $\tilde{\eta}$ is a $C^\infty$ form and $d\tilde{\eta} = 0$ in $\mathbb{R}^n$. $\eta - \tilde{\eta}$ is a form with compact support and $\omega = d(\eta - \tilde{\eta})$, which proves that $\omega$ is an exact form.

To show $H^k_c(\mathbb{R}^n) = \mathbb{R}$ it suffices to prove that if $\omega \in \Omega^k_c(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n} \omega = 0$ then $\omega = d\eta$ for some $\eta \in \Omega_{c}^{n-1}(\mathbb{R}^n)$. Take $D_R$ and $A$ as the previous case. We have $\omega = d\eta$ for some $(n-1)$-form $\eta$ and $d\eta = 0$ in $A$. On the other hand, by the stokes theorem, it follows that

$$0 = \int_{\mathbb{R}^n} \omega = \int_{D_R} \omega = \int_{D_R} d\eta = \int_{\partial D_R} \eta$$

By the previous corollary, we know that the restriction of $\eta$ on $\partial D_R$ is an exact form. As the radial projection is an isomorphism between the cohomology groups of $A$ and $\partial D_R$, we also have that $\eta$ is an exact form in $A$. Therefore $\eta = d\lambda$ in $A$ and, therefore, as before, we can find some $\tilde{\eta}$ such that $\omega = d(\eta - \tilde{\eta})$ where $\eta - \tilde{\eta}$ has compact support. So the cohomology class of $\omega$ vanishes in $H^k_c(\mathbb{R}^n)$. 
