1 barycentric subdivision

If $\sigma$ is a singular simplex, denote by $C_q(\sigma) \subset C_q(M)$ the subspace generated by $q$-simplexes of the form $\sigma \circ \ell$, where $\ell : \Delta_q \to \Delta_r$ is an affine simplex. As an affine simplex is determined by the image of its vertices we use the notation $\ell = [v_0, \ldots, v_q]$ where $v_j = \ell(e_j)$. One easily verifies that $\partial (\sigma \circ \ell) = \sigma \circ \partial \ell$ so the boundary operator maps $C_q(\sigma)$ to $C_{q-1}(\sigma)$ and therefore, the family of modules $C_q(\sigma)$ is a subcomplex.

The cone operator is the homomorphism $K_\sigma : C_q(\sigma) \to C_{q+1}(\sigma)$

defined by $\sigma \circ \ell \to \sigma \circ \ell'$, where $\ell' = [b, v_0, \ldots, v_q]$, and if $\ell = [v_0, \ldots, v_q]$ and $b = \frac{1}{r+1}(e_0 + \cdots + e_r)$ is the barycenter of the simplex $\Delta_r$. It is clear that, if $q > 0$ and $c = \sum a_i \sigma \circ \ell_i \in C_q(\sigma)$, then

$$\partial K_\sigma(c) + K_\sigma(\partial(c)) = c$$

and $\partial K_\sigma(c) + K_\sigma(\partial(c)) = c - \sum a_i \sigma(b)$. We deduce from this property that the homology group of the complex is trivial at dimension different from 0 and in dimension zero it coincides with the ring. A chain complex satisfying this property is called acyclic.

**Definition 1.1** *The barycentric subdivision, is the homomorphism $eta : C_r(M) \to C_r(M)$ defined inductively by the following conditions

1. $\beta(c) = c$ if $c \in C_0(M)$;
2. $\beta(\sigma) = K_\sigma \beta \partial \sigma$ if $\sigma : \Delta_r \to M$ is with $r > 0$.*

**Theorem 1.2** *For each $r$ the barycentric homomorphism,

$$\beta : C_r(M) \to C_r(M),$$

defined as above, satisfies the following properties:

1. $\beta \partial \sigma = \partial \beta(\sigma)$ for each singular simplex $\sigma : \Delta_r \to X$. Therefore $\beta$ is a morphism of the complex $(M)$;
2. For each $r \geq 0$ there exists a homomorphism $D_1 : C_r(M) \to C_{r+1}(M)$ such that

$$D_1 \partial \sigma + \partial D_1 \sigma = \sigma - \beta(\sigma)$$

for each $r$-singular simplex $\sigma : \Delta_r \to M$.***
**Proof.** The statement 1 is clear when $r = 0$, while both sides is equal to 0. Then for any $\sigma : \Delta_r \rightarrow X$, it follows:

$$
\partial \beta(\sigma) = \partial K_\sigma \beta(\partial \sigma) = \beta(\partial \sigma) - K_\sigma \beta(\partial \sigma) = \beta(\partial \sigma) - K_\sigma \beta(\partial \sigma) = \beta(\partial \sigma)
$$

where the penultimate step is by induction.

For 2, define the operator $D_1$ inductively, satisfying the following

a. $D_1(C_q(\sigma)) \subset C_{q+1}(\sigma)$,

b. $D_1(\sigma) = 0$ for $\sigma \in C_0(M)$,

c. $\partial D_1 c + D_1 \partial c = c - \beta(c)$ for each $c \in C_q(M)$.

Suppose, by induction, that we have defined $D_1$ for each $D_q(M)$ with $q < r$, satisfying the above conditions, Consider the following chain,

$$
z = \sigma - \beta(\sigma - D_1(\partial \sigma))
$$

It follows that

$$
\partial z = \partial \sigma - \partial \beta(\sigma) - \partial D_1(\partial \sigma) = \partial \sigma - \beta(\partial \sigma) - (\beta(\partial \sigma) + \partial \sigma - D_1(\partial^2 \sigma)) = \partial \sigma - \beta(\partial \sigma) + \beta(\partial \sigma) - \partial \sigma = 0
$$

Then $z \in C_r(\sigma)$ is a cycle. Since the complex $C(\sigma)$ is acyclic, there exists an element $D_1(\sigma) \in C_{r+1}(\sigma)$ such that $z = \partial D_1(\sigma)$. We can now extend $D_1$ by linearity to $C_r(M)$ and conclude by induction the proof of the theorem. 

**Corollary 1.3** The operator of barycentric subdivision $\beta$ induce the identity operator between each homology group.

Let $\mathcal{U} = \{U_1, \cdots, U_q\}$ be a family of subsets of a topological space $M$ whose interiors cover $M$. Let $C^\mathcal{U}_k(M) \subset C_k(M)$ be the subset generated by singular simplexes $\sigma : \Delta_r \rightarrow M$ such that the image $\sigma(\Delta_r)$ is contained in some $U_j \in \mathcal{U}$. Since $\partial(C^\mathcal{U}_k(M)) \subset C^\mathcal{U}_{k-1}(M)$ it follows that the subcomplexes whose homology groups are denoted by $H^\mathcal{U}_k(M)$ is such that the homomorphism $H^\mathcal{U}_k(M)$ is an isomorphism, due to the following,

**Theorem 1.4** There exist homomorphisms

$$
\Psi : C_r(M) \rightarrow C^\mathcal{U}_r(M) \subset C_r(M)
$$

and

$$
D : C_r(M) \rightarrow C_{r+1}(M)
$$

such that

1. $\partial \Phi = \Psi \partial$.

2. $\partial D(c) = D \partial(c) = c - \Phi(c)$ for each $c \in C_r(M)$. 

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3. $\Psi(c) = c$ for each $c \in C^d_r(M)$.

**Corollary 1.5** The homomorphisms of the homology groups induced by $\Psi$,

$$\Psi^*: H_k(M) \to H^d_k(M)$$

are isomorphisms.

**Proof.** (of the theorem) Let $\beta : C_r(M) \to C_r(M)$ be the homomorphism of the barycentric subdivision and $D_1 : C_r(M) \to C_{r+1}(M)$ is the homomorphism of theorem 1.2. Let

$$D_m : C_r(M) \to C_{r+1}(M)$$

be the homomorphism defined by $D_m = \sum_{i=0}^{m-1} D_1 \circ \beta^i$. Then it follows that

$$\partial D_m(c) + D_m \partial(c) = c - \beta^m(c)$$

Let $\sigma : \Delta_r \to M$ be a singular simplex. Let $\delta$ be the Lebesgue number of the cover of $\Delta_r$ by the pre-images of the interiors of $U_i$s. Then if $m$ is large enough, each affine subsimplex of $\Delta_r$ of the $m$-th barycentric subdivision of $\Delta_r$ has its diameter smaller than $\delta$, and therefore, is contained in one element of the cover. Thus, $\beta^m(\sigma) \in C^d_r(M)$. Let $m(\sigma)$ be the least integer with this property. Note that if $\tau$ is a face of $\sigma$ then $m(\tau) \leq m(\sigma)$. Define

$$D(\sigma) = D_{m(\sigma)}(\sigma)$$

and extend $D$ to $C_r(M)$ by linearity. As

$$\partial D_{m(\sigma)}(\sigma) + D_{m(\sigma)} \partial(\sigma) = \sigma - \beta^{m(\sigma)}(\sigma)$$

it follows

$$\partial D_{m(\sigma)}(\sigma) + \partial D \partial(\sigma) = \sigma - (\beta^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial(\sigma)) - D(\partial(\sigma)))$$

Finally define

$$\Psi(\sigma) = \beta^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial(\sigma)) - D(\partial(\sigma))$$

By extending $D$ be linearity we get,

$$\Psi : C_r(M) \to C_r(M)$$

thus

$$\partial D(c) + D \partial(c) = c - \Psi(c)$$

for each $c \in C_r(M)$. We still need to show property 1 and 3.

1. $\partial \Psi = \Psi \partial$

   Apply the equation 2 to $c = \partial(\sigma)$ we have

   $$\partial D(\partial(\sigma)) = \partial(\sigma) - \Psi(\partial(\sigma))$$

   On the other hand, apply the boundary operator on the same equation at $\sigma$ we have

   $$\partial D\sigma = \partial(\sigma) - \partial \Psi(\sigma)$$

   So we have

   $$\partial \Psi(\sigma) = \Psi(\sigma)$$

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for any singular simplex and, therefore, \( \partial \Psi(c) = \Psi(\partial c) \) for each singular chain, as we want to prove.

2. For \( c \in C^d_r(M) \), \( \Psi(c) = c \).

By linearity, it suffices to show that the implication for singular simplex \( \sigma \). In fact, if \( \sigma_i \) is the \( i \)-th face of the simplex \( \sigma \), then \( m(\sigma_i) \leq m(\sigma) \) and therefore \( m(\sigma_i) = 0 \), so \( D(\partial \sigma) - D_0(\partial \sigma) = 0 \) and therefore \( \Psi(\sigma) = \sigma \) as we claimed.

Finally, we check that if \( c \in C_r(M) \), then \( \Psi(c) \in C^d_r(M) \). Again, it suffices to show this when the chain is a unique singular simplex \( \sigma \). Since

\[
D_{m(\sigma)} \partial \sigma = \sum_{i=0}^{m(\sigma)-1} D_1 \circ \beta_i(\partial \sigma) = \sum_{i=0}^{m(\sigma)-1} \sum_{j=0}^{r} (-1)^j D_i \circ \beta^j(\sigma_j)
\]

and

\[
D(\partial \sigma) = \sum_{j=0}^{r} \sum_{i=1}^{m(\sigma)} D_i \circ \beta^j(\sigma_j)
\]

and \( m(\sigma_j) \leq m(\sigma) \), it follows that

\[
D_{m(\sigma)} \partial \sigma - D(\partial \sigma) = \sum_{j=0}^{r} \sum_{i=m(\sigma_j)}^{m(\sigma)-1} D_i \circ \beta^j(\sigma_j)
\]

if \( i \geq m(\sigma_j) \) then \( \beta^i(\sigma_j) \in C^d_q(M) \) and, since \( D_1(S^d_q(M)) \subset C^d_q(M) \) it follows that \( D_{m(\sigma)} \partial \sigma - D(\partial \sigma) \in C^d_q(M) \) which conclude the proof.

**Theorem 1.6 (Meyer-Vietoris)** If \( M = \text{Int}(U) \cap \text{Int}(V) \), then, for each \( r \) there exists a homomorphism \( \delta_r : H_r(M) \to H_{r-1}(U \cap V) \) such that the following Meyer-Vietoris sequence is exact:

\[
\cdots \xrightarrow{\delta_{k+1}} H_k(U \cap V) \xrightarrow{\alpha_k} H_k(U) \oplus H_k(V) \xrightarrow{\beta_k} H_k(M) \xrightarrow{\delta_k} H_{k-1}(U \cap V) \cdots
\]

**Proof.** Let \( \mathcal{U} = \{U,V\} \) and it follows from the isomorphism between \( H^d_k(M) \) and \( H_k(M) \) and the exactness of the following sequence of chain complexes

\[
0 \to C^d_q(U \cap V) \to C^d_q(U) \oplus C^d_q(V) \to C^d_q(M) \to 0
\]

while the first homomorphism is \( c \mapsto (c,c) \) and the second is \( (c_1,c_2) \mapsto c_1 - c_2 \).

**Corollary 1.7** \( H_k(S^n) = A \) if \( k = 0, n \) and \( H_k(S^n) = 0 \) if \( k \neq 0, n \).

**Proof.** Suppose \( U = S^1 \setminus \{p\}, V = S^1 \setminus \{q\} \) where \( p,q \) are different points in \( S^1 \). Then we have, from the Meyer-Vietoris sequence,

\[
H_1(U) \oplus H_1(V) \xrightarrow{i_*} H_1(S^1) \xrightarrow{\partial} H_0(U \cap V) \xrightarrow{j_*} H_0(U) \oplus H_0(V)
\]

Since \( H_1(U) = H_1(V) = 0 \), we have that \( \partial \) is injective, so it is onto its image in \( H_0(U \cap V) \), which is the kernel of \( i_* \). On the other hand, \( H_0(U \cap V) = \mathbb{A}^2 \) because it contains two connected components. Any 0-chain in \( U \cap V \) has the form \( z = x + y \), where \( x \) and \( y \) are 0-chains in each of the two components.

The homomorphism is given by \( i_*(\alpha, \beta) = (\alpha + \beta, \alpha + \beta) \). Consequently, its kernel is of the form \( (\alpha, -\alpha) \) with \( \alpha \in A \), which is isomorphic with \( A \). So, \( H_1(S^1) = A \).
Similarly, for $S^n$, suppose $U = S^n \setminus \{p\}$ and $V = S^n \setminus \{q\}$ for different $p, q \in S^n$. Then $U \cap V$ is homotopic with $S^{n-1}$, and we have

$$H_r(U) \oplus H_r(V) \to H_r(S^n) \to H_{r-1}(U \cap V) \to H_{r-1}(U) \oplus H_{r-1}(V)$$

which reduces to

$$0 \to H_r(S^n) \to H_{r-1}(S^{n-1}) \to 0$$

using induction, we can complete the proof.

\[\blackbox\]

**Theorem 1.8 (excision theorem)** Z ⊂ Y ⊂ X such that Z ⊂ Int(Y), then the map of inclusion

$$(X \setminus Z, Y \setminus Z) \hookrightarrow (X, Y)$$

induces isomorphisms between homology groups.

**Proof.** Let $U = X \setminus Z$ and $V = Y$. From the previous theorems, we have that the map $C^d_k(X) \xrightarrow{\Psi} C_k(X)$ induces the isomorphisms between $H_k(X)$ and $H^d_k(X)$ for each $k$. Then we imply the five lemma to the long exact sequence and conclude that $H_k(X, Y)$ is isomorphic with $H^d_k(X, Y)$. It suffices to prove that $H^d_k(X, Y)$ is isomorphic with $H_k(X - Z, Y - Z)$.

For this, consider the inclusion

$$i_* : C_p(X - Z) \to C^d_p(X)\over C^d_p(Y)$$

which is induced by the inclusion, since $i(C_p(X - Z)) \subset C^d_p(X)$. If $c_p$ is a chain of $C^d_p(X)$, then each term of it belongs to either $C_p(X - Z)$, or $C_p(Y)$. So its equivalence class $c_p + C^d_p(Y) = c'_p + C^d_p(Y)$ where $c'_p$ consists only the terms taking values in $X - Z$, so $c'_p \in C_p(X - Z)$ and the map $i_*$ is thus surjective.

The kernel of this map $i_*$ is as follows

$$C_p(X - Z) \cap C^d_p(Y) = C_p(Y - Z).$$

So $C^d_p(X - Z)\over C_p(Y - Z)$ and $C^d_p(X)\over C^d_p(Y)$ is isomorphic, thus completing the proof.

\[\blackbox\]

Given a subset $Y$ of a topological space $X$, consider the equivalence relation which identifies two distinct points if and only if they belong to $Y$. The space of equivalence classes is denoted by $X/Y$. Let $q : X \to X/Y$ be the quotient map.

**Corollary 1.9** If there exists a neighborhood $V \subset X$ of $Y$ such that $Y$ is a retraction then the quotient map induces isomorphisms

$$q_* : H_k(X, Y) \to H_k(X/Y, Y/Y)$$

**Proof.** Consider the commutative graph as follows

$$\begin{array}{ccc}
H_r(X, Y) & \to & H_r(X, V) \\
\downarrow & & \downarrow \\
H_r(X/Y, Y/Y) & \to & H_r(X/Y, V/Y) \\
\end{array}$$

The right-hand vertical map, $q_*$, is an isomorphism, since when restricted on on the complement of $Y$, $q$ is an homeomorphism.
Now observe the upper left horizontal map. In the exact sequence
\[ \cdots \rightarrow H_k(V,Y) \rightarrow H_k(X,Y) \rightarrow H_k(X,V) \rightarrow H_{k-1}(V,Y) \rightarrow \cdots \]
we note that \( V \) can be retracted to \( Y \) then \( H_k(V,Y) = 0 \) for any \( k \). Thus we get the isomorphism between \( H_k(X,Y) \) and \( H_k(X,V) \). Similarly the lower left homomorphism is isomorphism, too. By the excision theorem, we have the upper right and lower right homomorphisms are all isomorphisms. Then since the diagram commutes, we have finished the proof.

**Corollary 1.10** If \( f : M \rightarrow \mathbb{R} \) is a Morse function with a unique critical point of index \( \lambda \) in the compact set \( f^{-1}([a,b]) \) then \( H_k(M^b,M^a) = A \) if \( k = \lambda \) and is equal to zero if \( k = \lambda \).

**Proof.**
\[
H_k(M^b,M^a) = H_k(M^a \cup_e e^\lambda, M^a) = H_k(e^\lambda, \partial e^\lambda) = H_k(S^\lambda) = \begin{cases} \mathbb{R}, & \text{if } k = 0, \lambda \\ 0 & \text{if } k \neq 0, \lambda \end{cases}
\]

**Definition 1.11** For a non-negative integer \( k \), the \( k \)th Betti number \( b_k(X) \) of the space \( X \) is defined as the rank of the abelian group \( H_k(X) \), the \( k \)th homology group of \( X \).

We have computed that
\[
b_k(M^b,M^a) = \begin{cases} 1, & \text{if } k = 0, \lambda; \\ 0, & \text{if } k \neq 0, \lambda. \end{cases}
\]

## 2 Morse inequalities

**Lemma 2.1** Let \( Z \subset Y \subset X \) be topological spaces such that
1. \( b_\lambda(X,Y) = \dim H_\lambda(X,Y) < \infty \),
2. \( b_\lambda(X,Z) = \dim H_\lambda(X,Z) < \infty \)
3. \( b_\lambda(Y,Z) = \dim H_\lambda(Y,Z) < \infty \)
4. \( b_\lambda(X,Y) = b_\lambda(X,Z) = b_\lambda(Y,Y) = 0 \) if \( \lambda \geq n \). Then we have:
   a. \( b_\lambda(X,Z) \leq b_\lambda(X,Y) + b_\lambda(Y,Y) \)
   b. \( \sum_{\lambda=0}^{n} (-1)^\lambda b_\lambda(X,Z) = \sum_{\lambda=0}^{n} (-1)^\lambda (b_\lambda(X,Y) + b_\lambda(Y,Z)) \)

**Proof.** Consider the following exact sequence:
\[
\cdots \xrightarrow{\partial_{\lambda+1}} H_\lambda(Y,Z) \xrightarrow{i_*} H_\lambda(X,Z) \xrightarrow{j_*} H_\lambda(X,Y) \xrightarrow{\partial_\lambda} H_{\lambda-1}(Y,Z) \cdots
\]
Since the dimension of $H_\lambda(X, Z)$ is the sum of the dimension of the kernel of the operator $j_\lambda$ and the $\text{rank}(i_\lambda)$, from the exactness, we have
\[ b_\lambda(X, Z) = \text{rank}(i_\lambda) + \text{rank}(j_\lambda) \leq b_\lambda(X, Y) + b_\lambda(Y, Y) \]
Thus
\[
\sum_{\lambda=0}^{n} (-1)^{\lambda} b_\lambda(X, Z) \\
= \sum_{\lambda=0}^{n} (-1)^{\lambda} (\text{rank}(i_\lambda) + \text{rank}(j_\lambda)) \\
= \sum_{\lambda=0}^{n} (-1)^{\lambda} (b_\lambda(Y, Z) - \text{rank}(\partial_{\lambda+1}) + b_\lambda(X, Y) - \text{rank}(\partial_\lambda)) \\
= \sum_{\lambda=0}^{n} (-1)^{\lambda} (b_\lambda(Y, Z) + b_\lambda(X, Y)) + \sum_{\lambda=0}^{n} (-1)^{\lambda} (\text{rank}(\partial_{\lambda+1}) - \text{rank}(\partial_\lambda)) \\
= \sum_{\lambda=0}^{n} (-1)^{\lambda} (b_\lambda(Y, Z) + b_\lambda(X, Y))
\]

**Lemma 2.2** If $S$ is a subadditive function, i.e., for $X_{i-1} \subset X_i \subset X_{i+1}$, $S(X_{i+1}, X_{i-1}) \leq S(X_{i+1}, X_{i}) + S(X_{i}, X_{i-1})$, and suppose
\[ X_0 \subset X_1 \subset \cdots \subset X_n, \]
then
\[ S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1}) \]
If $S$ is additive, then
\[ S(X_n, X_0) = \sum_{i=1}^{n} S(X_i, X_{i-1}) \]

**Proof.** By induction. □

**Lemma 2.3** The function
\[ S_\lambda(X, Y) = b_\lambda(X, Y) - b_{\lambda-1}(X, Y) + b_{\lambda-2}(X, Y) - \cdots \pm b_0(X, Y) \]
is subadditive.

**Proof.** The exact sequence,
\[ \cdots \xrightarrow{\partial_{\lambda+1}} H_\lambda(Y, Z) \xrightarrow{i_\lambda} H_\lambda(X, Z) \xrightarrow{j_\lambda} H_\lambda(X, Y) \xrightarrow{\partial_\lambda} H_{\lambda-1}(Y, Z) \]
implies that
\[
0 \leq \text{rank}(\partial_{\lambda+1}) \\
= b_\lambda(Y, Z) - \text{rank}(i_\lambda) \\
= b_\lambda(Y, Z) - b_\lambda(X, Z) + \text{rank}(j_\lambda) \\
= b_\lambda(Y, Z) - b_\lambda(X, Z) + b_\lambda(X, Y) - \text{rank}(\partial_\lambda) \\
= b_\lambda(Y, Z) - b_\lambda(X, Z) + b_\lambda(X, Y) \\
- b_{\lambda-1}(Y, Z) + b_{\lambda-1}(X, Z) - b_{\lambda-1}(X, Y) + \text{rank}(\partial_{\lambda-1})
\]
continuing writing this in the same fashion, we have
\[ X_\lambda(Y, Z) - S_\lambda(X, Z) + S_\lambda(X, Y). \]

\[ \text{Theorem 2.4} \]
Let \( f : M \to \mathbb{R} \) be a Morse function on a compact manifold \( M \). Let
1. \( c_\lambda \) is the number of points of critical points of index \( \lambda \).
2. \( b_\lambda(M) \) is the dimension of the vector space \( H_\lambda(M, K) \), where \( K \) is a field.
Then
a. \( b_\lambda - b_{\lambda-1} + \cdots + b_0 \leq c_\lambda - c_{\lambda-1} + \cdots + c_0 \).
b. \( \sum_{\lambda=0}^{m} (-1)^\lambda b_\lambda = \sum_{\lambda=0}^{m} (-1)^\lambda c_\lambda \).

\[ \text{Proof.} \]
We can suppose, by a small perturbation if necessary, that each level set can possibly contain only one critical point. Then suppose \( a_1 < \cdots < a_k \) are regular values of \( f \) such that \( M^{a_i} \setminus M^{a_i-1} \) contains one critical point of index \( \lambda_i \). By the excision theorem, \( H_s(M^{a_i-1} \cup \mathbb{e}^{\lambda_i}, M^{a_i-1}) \) is isomorphic to \( H_s(e^{\lambda_i}, \partial e^{\lambda_i}) \) which is isomorphic to \( \mathbb{R} \) if \( s = \lambda_i \) and is equal to 0 if \( s \neq \lambda_i \).
Since,
\[
\emptyset = M^{a_0} \subset M^{a_1} \subset \cdots \subset M^{a_k} = M
\]
and by the previous lemmas, \( S_\lambda(M, \emptyset) \leq \sum_{i=0}^{k} S_\lambda(M^{a_i}, M^{a_i-1}) \), we conclude the proof of the theorem.

3 CW-complex

\[ \text{Definition 3.1} \]
A CW-complex is a topological space \( M \) such that
\[ M_0 \] is a discrete set.
2. \( M_{n-1} \subset M_n \) are closed subsets.
3. for each \( n \in \mathbb{N} \), there exists a family of continuous functions,
\[ \Phi^n_\alpha : \mathbb{B}^n \to M_n \subset M \]
such that
a. \( \Phi^n_\alpha(S^{n-1}) \subset M_{n-1} \) really where \( \Phi^n_\alpha \) means \( \Phi^n_\alpha|_{\partial B^n} \);
b. \( \Phi^n_\alpha|_{B^n} \) is a homeomorphism onto its image;
c. \( \Phi^n_\alpha(B^n) \cap \Phi^n_\beta(B^n) = \emptyset \) if \( \alpha \neq \beta \);
d. \( M_n \setminus M_{n-1} = \cup_\alpha \Phi^n_\alpha(B^n) \).
4. \( F \subset M \) is closed if and only if \( (\Phi^n_\alpha)^{-1}(F) \) is closed for all \( \alpha \) and \( n \in \mathbb{N} \).

\( M_n \) is called the skeleton of dimension \( n \) of \( M \), \( e^n_\alpha = \Phi^n_\alpha(B^n) \) are cellars of dimension \( n \), \( \Phi^n_\alpha \) are characteristic functions.