Differential Topology notes, 15

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Given two oriented compact manifolds \( M, N \) of the same dimension, \( f : M \to N \) is continuous, we have defined \( \text{deg}(f) = \sum_{x \in f^{-1}(y)} \text{sign}(x) \in \mathbb{Z} \), where \( y \) is a regular value. Till now we have proved the following properties.

1. If \( f \) and \( g \) are homotopic, then \( \text{deg}(f) = \text{deg}(g) \).
2. If \( M^n = \partial W^{n+1}, F : W \to N \) is continuous, then \( \text{deg}(F|_M) = 0 \).
3. If \( F : M^n \to S^n \) has degree zero, then we can extend it continuously to \( F : W \to S^n \).

We make two further observations: One, suppose \( M \) is not compact. If there exists a proper homotopy of two proper continuous maps \( f, g \) between two oriented manifolds then the maps have the same degree. Two, for manifolds which are not orientable, we can define the notion of the degree module 2. For a \( C^\infty \) function, this notion is just the number of pre-images of a regular value.

## 1 Indices of singularities of a vector field

**Definition 1.1** Let \( X : U \subset \mathbb{R}^n \to \mathbb{R}^n \) be a continuous vector field and \( x_0 \in U \) is an isolated singularity of \( X \), i.e., \( X(x_0) = 0 \) and there exists some \( \sigma > 0 \) such that \( X(x_0 + x) \neq 0 \) for \( 0 < \|x\| \leq \sigma \). Define the index of \( X \) and \( x_0 \), \( \text{Ind}(X, x_0) \) as the degree of the map

\[
S^{n-1} \to S^{n-1} : x \mapsto \frac{X(x_0 + \varepsilon x)}{\|X(x_0 + \varepsilon x)\|}
\]

**Definition 1.2** \( x_0 \in U \) is called a hyperbolic singularity of a \( C^1 \) vector field, \( X : U \to \mathbb{R}^n \) if \( DX(x_0) \) does not have eigenvalues in the imaginary axis. The stable subspace of \( X \) at \( x_0 \) is the eigenspace \( E^s_{x_0} \) associated with the eigenvalues with real part \( < 0 \).

**Propostion 1.3** If \( x_0 \) is a hyperbolic singularity of a \( C^1 \) vector field \( X : U \subset \mathbb{R}^n \to \mathbb{R}^n \) then

\[
\text{Ind}(X, x_0) = (-1)^{\dim E^s_{x_0}}
\]

**Proof.** Let \( A_0 = DX(x_0) \). For \( \epsilon > 0 \) smaller enough and \( x \in S^{n-1} \), it follows that \( X(x_0 + \varepsilon x) = A_0(\varepsilon x) + r(\varepsilon x) \) with \( \frac{r(\varepsilon x)}{\|X(x_0 + \varepsilon x)\|} \to 0 \) when \( \varepsilon \to 0 \). On the other hand, as \( A_0 \) does not have zero eigenvalue if follows that \( \|A_0(x)\| \geq m \geq 0 \) for any \( x \in S^{n-1} \). Thus, it follows \( \|A_0(x) + \frac{1}{\varepsilon} sr(x)\| \neq 0 \) for all \( s \in [0,1] \) and small enough \( \epsilon \). Therefore, the map

\[
S^n \to S^n, x \mapsto \frac{A_0(x) + \frac{1}{\varepsilon} sr(x)}{\|A_0(x) + \frac{1}{\varepsilon} sr(x)\|}
\]

is a homotopy and thus

\[
\text{Ind}(X, x_0) = \text{Ind}(A_0, 0)
\]

To compute the right hand side, we have, by the lemma from previous notes, that there exists a homotopy \( A_t \), which is homotopic to the identity if the number of eigenvalues with real part \( < 0 \) is even and is homotopic to the map \( (x_1, \ldots, x_n) \mapsto (-x_1, x_2 \ldots, x_n) \) if this number is odd. Then \( \text{Ind}(A_0, 0) \) is equal to 1 if the eigenspace corresponding to the eigenvalues with negative real parts has even dimension and equal to \( -1 \) otherwise.
Corollary 1.4 If \( x_0 \in U \) is a hyperbolic singularity of the vector field \( X : U \to \mathbb{R}^n \) and \( \varphi : U \to V \subset \mathbb{R}^n \) is a \( C^\infty \) diffeomorphism, \( Y : V \to \mathbb{R}^n \) is the field \( \varphi_* X : y \mapsto (D\varphi)_{\varphi^{-1}(y)} X(\varphi^{-1}(y)) \). Then \( \varphi(x_0) \) is a hyperbolic singularity of \( Y \), and,
\[
\text{Ind}(X, x_0) = \text{Ind}(Y, \varphi(x_0)).
\]

**Proof.** Let \( y_0 = \varphi(x_0) \), then
\[
DY_{y_0} = D\varphi_{x_0} \cdot DX_{x_0} \cdot D\varphi^{-1}_{x_0}
\]
So \( DX_{x_0} \) and \( DY_{y_0} \) have the same spectrum and thus \( \dim(E^s_{x_0}) = \dim(E^s_{y_0}) \).

Lemma 1.5 Let \( x_0 \in U \) be an isolated singularity of the continuous vector field \( X : U \subset \mathbb{R}^n \to \mathbb{R}^n \). If \( \varphi : U \to V \subset \mathbb{R}^n \) is a \( C^\infty \) diffeomorphism and \( Y = \varphi_* X \), then
\[
\text{Ind}(X, x_0) = \text{Ind}(Y, y_0), \quad y_0 = \varphi(x_0)
\]

**Proof.** Let \( a > 0 \) such that
\[
0 < \|x - x_0\| \leq a \Rightarrow X(x) \neq 0
\]
\[
0 < \|y - y_0\| \leq a \Rightarrow Y(y) \neq 0
\]
Let \( b < \frac{a}{2} \) be small enough such that
\[
\|x - x_0\| < b \Rightarrow \|\varphi(x) - y_0\| < \frac{a}{2}
\]
Let \( \epsilon > 0 \) and \( \tilde{X}(x) \) be a \( C^\infty \) vector field whose singularities are all hyperbolic and
\[
\|\tilde{X}(x) - X(x)\| \leq \epsilon, \quad \forall x \in U.
\]
If \( \epsilon > 0 \) is small enough \( \varphi_* \tilde{X} \) is very close to \( Y \) such that the singularities of \( \varphi_* \tilde{X} \) is contained in the ball of radius \( \frac{\epsilon}{2} \) and center \( y_0 \) and the maps \( S^{n-1} \to S^{n-1} \),
\[
y \mapsto \frac{Y(y_0 + ay)}{\|Y(y_0 + ay)\|} \quad \text{and} \quad y \mapsto \frac{\varphi_* \tilde{X}(y_0 + ay)}{\|\varphi_* \tilde{X}(y_0 + ay)\|}
\]
are very close and therefore are homotopic and has thus the same degree.

For \( \epsilon \) small enough, the singularities of \( \tilde{X} \) are contained in the ball of radius \( \frac{\epsilon}{2} \) and center \( x_0 \). By the previous corollary, the sum of the indices of the singularities of \( \tilde{X} \) in \( B(x_0, a) \) is equal to the sum of the indices of the singularities of \( \varphi_* \tilde{X} \) in \( B(x_0, a) \). Let \( \delta \) small enough such that the balls with radius \( \delta \) and centers \( x_1, x_2, \cdots, x_k \) those singularities are pairwise disjoint and is contained in \( B(x_0, a) \).

Let \( W = B(x_0, a) \setminus \bigcup_i \overline{B_{\delta}(x_i)} \) where \( x_1, x_2, \cdots, x_k \) are singularities of \( \tilde{X} \) in \( B(x_0, a) \). Since \( \tilde{X} \) does not has singularities in \( W \), we can define the continuous function from \( W \) to \( S^{n-1} \) as
\[
x \mapsto \frac{\tilde{X}(x)}{\|\tilde{X}(x)\|}
\]
Therefore, by the theorem from previous notes, the restriction of this function on \( \partial W \) has degree zero. Thus, we conclude with
\[
\text{Ind}(X, x_0) = \sum_{i=1}^k \text{Ind}(\tilde{X}, x_i)
\]
Similarly, we have
\[ \text{Ind}(Y, y_0) = \sum_{i=1}^{k} \text{Ind}(\varphi_* \tilde{X}, \varphi(x_i)) \]
and finally, we have \( \text{Ind}(X, x_0) = \text{Ind}(Y, y_0) \).

**Definition 1.6** If \( X : M \to TM \) is a \( C^0 \) vector field and \( x_0 \in M \) is an isolated singularity of \( X \), define
\[ \text{Ind}(X, x_0) = \text{Ind}(\varphi_* X, \varphi(x_0)) \]
where \( \varphi : U \subset M \to \tilde{U} \subset \mathbb{R}^m \) is a local chart.

**Theorem 1.7** Let \( M \) be compact. If \( Y, X : M \to TM \) are vector fields whose singularities are all isolated. Then
\[ \sum_{X(x) = 0} \text{Ind}(X, x) = \sum_{Y(y) = 0} \text{Ind}(Y, y) \]
This number is called the Euler characteristic of \( M \) and is denoted as \( \chi(M) \).

**Proof.** By approximating the vector fields with \( C^\infty \) fields with all singularities hyperbolic, we can suppose \( X \) and \( Y \) have this property. As we have seen, there exists continuous vector fields \( X_t, t \in [0, 1], X_0 = X, X_1 = Y \) such that all the singularities of \( X_t \) are isolated. To find this \( X_t \), we can construct, locally, using coordinates, \( (X_i)_t = tX^i + (1 - t)Y^i \) to change piecewise the vector field from \( X \) to \( Y \). (check this!). By the invariance of the degree under homotopy we can find that \( 0 = t_0 < t_1 < \cdots < t_\ell = 1 \), such that
\[ \sum_{X_t (x) = 0} \text{Ind}(X_t, x) = \sum_{X_{t_\ell} (x) = 0} \text{Ind}(X_{t_\ell}, x), j = 0, 1, \ldots, \ell - 1 \]
thus the conclusion.

**Proposition 1.8** If \( M \) is a manifold of odd dimension, then \( \chi(M) = 0 \).

**Proof.** Suppose \( X : M \to TM \) is a vector field with all singularities hyperbolic. Denote these singularities as \( x_1, x_2, \cdots, x_k \) and we have, by previously theorems, \( \chi(M) = \sum_{i=1}^{k} (-1)^{\dim E^s_{x_i}} \). The vector field \(-X\) has the same singularities, but the stable subspace of \(-X\) for each singularity \( x_i \) is the unstable singularity of \( X \) at singularity \( x_i \), thus,
\[ \chi(M) = \sum_{i} \text{Ind}(-X, x_i) = \sum_{i} (-1)^{n - \dim E^s_{x_i}} = -\chi(M) \]
so, \( \chi(M) = 0 \).

**Theorem 1.9** Let \( M \) be a compact manifold. Then there exists a \( C^\infty \) vector field in \( M \) whose singularities are all hyperbolic and has the same index.

**Proof.** Let \( X \) be a \( C^\infty \) vector field in \( M \) whose singularities are all hyperbolic. Let \( x, y \in M \) be singularities such that \( \text{Ind}(X, x) = -\text{Ind}(X, y) \).

Using an embedding \( C^\infty \) arc connecting \( x \) and \( y \), which is disjoint with all the other singularities, we can find the corresponding flow and construct the embedding (tubular flow)
\[ \varphi : (-\epsilon, 1 + \epsilon) \times D^{n-1} \to M \]
such that $\varphi(\epsilon, 1+\epsilon) \times D^{n-1}$ intersects with the set of singularities of $X$ only in points $x = \varphi(0,0)$ and $y = \varphi(1,0)$. Let $Y : (\epsilon, 1+\epsilon) \times D^{n-1} \to \mathbb{R}^n$ be the vector field such that $\varphi_* Y = X$. We take a neighborhood $B$ of $[0,1] \times \{0\}$ which is diffeomorphic to a ball whose boundary is diffeomorphic with a sphere and $B_0, B_1$ are balls centered at $(0,0)$ and $(1,0)$ whose closure is contained in $B$.

Let $W = B \setminus B_0 \cup B_1$, then the map

$$\partial W \to S^{n-1}, x \mapsto \frac{Y(x)}{\|Y(x)\|}$$

can be extended to $W$, and, therefore, has degree zero. But as $\text{Ind}(Y, (0,1)) = -\text{Ind}(Y, (1,0))$ it follows that the map

$$\partial B \to S^{n-1}, x \mapsto \frac{Y(x)}{\|Y(x)\|}$$

has degree zero. Then we can extend it differentially to $B$, which, is indeed a vector field without any singularities. Therefore, $Y|_{(\epsilon, 1+\epsilon) \times D^{n-1} \setminus B}$ can be extended to a $\tilde{Y}$ which has no singularities. Thus the vector field $Z$ which coincides with $X$ out of $\varphi((\epsilon, 1+\epsilon) \times D^{n-1})$ and coincides with $\varphi_* \tilde{Y}$ in $\varphi((\epsilon, 1+\epsilon) \times D^{n-1})$ has the number of pairs of singularities with distinct indices strictly smaller that of $X$.

Continuing with the same trick, we can finally find a vector field in which all the singularities have the same index.

**Corollary 1.10** If $\chi(M) = 0$ then there exists a vector field in $M$ without singularities.

**Corollary 1.11** Any vector field in $S^2$ has at least one singularity. (Hairy ball theorem)

**Propostion 1.12** If a manifold of even dimension, $M$, is the connected sum of two manifolds $M_1$ and $M_2$, then $\chi(M) = \chi(M_1) + \chi(M_2) - 2$.

**Proof.** Let $B_j \subset M_j$ be two embedding balls and $\phi_j : M_j \setminus B_j \to M$ are embeddings such that $M$ is the union of the images of $\phi_1$ and $\phi_2$ and the intersection of the images is an embedding sphere $S \subset M$. In $M_1$ construct a vector field which has a unique singularity in $B_1$, hyperbolic and attractive, and is transversal to $\partial B_1$. Suppose that all other singularities of $X_1$ are hyperbolic. Similarly, we can construct a vector field $X_2$ which has in $B_2$ a unique singularity, hyperbolic and repulsive, and is transversal to $\partial B_2$.

Now suppose $x_1, x_2, \ldots, x_k$ are singularities of $X_1$ in $M_1 \setminus B_1$, and $y_1, y_2, \ldots, y_\ell$ are singularities of $X_2$ in $M_2 \setminus B_2$. So

$$\sum_{i=1}^k \text{Ind}(X_1, x_i) = \chi(M_1) - 1$$

$$\sum_{j=1}^\ell \text{Ind}(X_2, y_j) = \chi(M_2) - 1$$

We can then construct a vector field $X$ on $M$ which is transversal to $S$, and such that $\phi_1^*(X)$ coincides with $X_1$ out of a small neighborhood of the boundary where the the field does not vanish. Thus,

$$\sum_{x \in M_1, X(x) = 0} \text{Ind}(X, x) = \sum_{x \in M_1 \setminus B_1, X_1(x) = 0} \text{Ind}(X_1, x)$$

$$\sum_{x \in M_2, X(y) = 0} \text{Ind}(X, y) = \sum_{y \in M_2 \setminus B_2, X_1(y) = 0} \text{Ind}(X_2, x)$$

So, by definition, it follows $\chi(M) = \chi(M_1) + \chi(M_2) - 2$. 

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**Definition 1.13** Let $W$ be a manifold with boundary. $\chi(W)$ is defined as the sum of the indices of the singularities of a vector field which is transversal to the boundary of $W$, and at the boundary points to the inside of the manifold. (or, points to the outside of the manifold)

Then we have the following corollary.

**Corollary 1.14** $W_1, W_2$ are two manifolds with boundaries of the same dimension. $f : \partial W_1 \to \partial W_2$ is a diffeomorphism and let $M = W_1 \cup_f W_2$, then it follows that $\chi(M) = \chi(W_1) + \chi(W_2)$.

## 2 Number of intersections

Just as the notion of transversality of a map with respect to a submanifold is a generalization of the notion of regular value, the definition below generalize the definition of topological degree.

**Definition 2.1** Let $M$ and $N$ be oriented manifolds, $M$ is compact and $S \subset N$ is a closed submanifold and is oriented such that $\dim(M) + \dim(S) = \dim(M)$. If $f : M \to N$ is a $C^r$ map, which is transversal to $S$ then the number of intersection of $f$ with $S$ is defined as

$$ \sharp f \cap S = \sum_{x \in f^{-1}(S)} \text{sign}(x) $$

where $\text{sign}(x) = 1$ if the image under $Df_x$ of a positive base of $TM_x$ followed by a positive base of $TS_{f(x)}$ is a positive base of $TN_{f(x)}$ and $\text{sign}(x) = -1$ otherwise.

Using very similar arguments with that in the topological degree, we can prove that the number of intersections is an invariant under homotopy and therefore can be defined for continuous functions, too.

If $M$ is a oriented compact manifold, the sum of the indices of the (hyperbolic) singularities of a vector field coincides with the number of intersections of $X$ with the null section of $TM$, that is,

$$ \sharp X \cap S_0 = \chi(M). $$

As the space of vector fields is a vector space, two vector fields are always homotopic: $t \mapsto (1 - t)X + tY$. So, the invariance under homotopy of the number of indices of singularities gives another proof of the fact that the sum of the indices of the singularities does not depend on the choice of the vector field.