1 Transversality theorems

**Definition 1.1** A differentiable map $f : M \to N$ is transversal to a submanifold $S \subset N$ if for any $x \in M$ we have that either $f(x) \notin S$ or $Df_x(TM_x) + TS_{f(x)} = TN_{f(x)}$. If $S_1 \subset N$ is another submanifold, we say that $S_1$ is transversal to $S$ and write $S_1 \cap S$ if the inclusion $i : S_1 \to N$ is transversal to $S$ i.e., for any $y \in S_1 \cap S, T(S_1)_y + TS_y = TN_y$.

**Proposition 1.2** If $f$ is a $C^r$ map, $r \geq 1$, which is transversal to $S$. Then $f^{-1}(S)$ is either empty or a submanifold of $M$ of the same codimension of $S$. In particular, if
\[
\text{cod}(S) = \dim N - \dim S > \dim M
\]
then $f^{-1}(S)$ is empty.

**Proof.** (sketch) First show that when $S$ is a submanifold of $N$ and let $f : M \to N$ be a differentiable map, and suppose that at each point $p \in f^{-1}(S)$, $Df_p$ is surjective, i.e., the map $f$ at point $p$ is submersion. Under these conditions, we claim that $f^{-1}(S)$ is a submanifold of $M$, whose codimension is the same with the codimension of $S$ in $N$.

Then, suppose that $f^{-1}(S)$ is not empty, it suffices to prove that $Df_p$ is surjective for each $p \in f^{-1}(S)$. This is equivalent with the definition of the transversality.

**Lemma 1.3** If $f \in C^r(M, N)$ is transversal to $S$ and $S$ is a closed submanifold of class $C^\infty$ of $N$ then for any $x \in M$ there exists $\varepsilon_x > 0$ and a neighborhood $V_x \subset M$ such that if $g \in C^r(M, N)$ and $d(j^1g(y), j^1f(y)) < \varepsilon_x$ for any $y \in V_x$ then the restriction of $g$ on $V_x$ is transversal to $S$.

**Proof.** Omitted.

**Theorem 1.4** If $S$ is a closed submanifold, then the set of transformations $f : M \to N$ of class $C^r, r \geq 1$ which are transversal to $S$ is open in $C^r(M, N)$.

**Proof.** It follows from the above lemma.

**Remark** The conclusion of false if without the conditions that $S$ is closed. An example to show this is as follows. In $\mathbb{R}^3 \supset S = \{(x, y, z); x = y = 0, z \in (0, 1)\}$, let $f : \mathbb{R} \to \mathbb{R}^3, x \mapsto (0, 0, x)$. So $f \cap S$ since $f^{-1}(S)$ is empty. But in any neighborhood of $f$ there is some $g$ such that $g^{-1}(S) \neq \emptyset$. But $g$ can not be transversal to $S$ because $\dim(\mathbb{R}) + \dim(S) < \dim(\mathbb{R}^3)$.

**Lemma 1.5** If $F : M \times P \to N$ is a $C^\infty$ map and $S \subset N$ is a submanifold. For each $y \in P$ let $F_y : M \to N$ be a map $F_y(x) = F(x, y)$. Suppose $F$ is transversal to $S$, if $y$ is a regular value of the restriction of the projection $\pi_2 : M \times P \to P$ on $F^{-1}(S)$, then $F_y$ is transversal to $S$.

**Proof.** Let $x \in M$ such that $F_y(x) \in S$. As $y$ is a regular value, if follows that there exists a subspace $E_1 \subset T(F^{-1}(S))_{(x, y)}$ such that the restriction of $(D\pi_2)_{(x, y)}$ on $E_1$ is an isomorphism. If
follows that $T(F^{-1}(S))_{(x,y)} = E_1 \oplus E_2$ where $E_2$ is a subspace contained in the kernel of $(DF_x)_{(x,y)}$. Let $E_3$ be the complement subspace of $E_2$ in the the kernel of $(DF_x)_{(x,y)}$. Since the derivative $DF_{(x,y)}$ maps $E_1 \oplus E_2$ to the tangent space of $N$ at point $f(x,y)$, namely, $TN_{F(x,y)}$, and since $F$ is transversal, so the image of $E_3$ is a subspace $E_4 \subset TN_{F(x,y)}$ such that $TN_{F(x,y)} = TS_{F(x,y)} \oplus E_4$.

Now, we have $DF_y(TM_x) = DF_{(x,y)}(E_2 \oplus E_3) \supset DF_{(x,y)} = E_4 = TN_{F(x,y)} \oplus TS_{F(x,y)}$, it follows that $F_y$ is transversal at $x$.

**Lemma 1.6** If $F : M \to N$ is of class $C^\infty$ then the set of regular values of $F$ is a residual subset of $N$.

**Proof.** Let $\phi_i : W_i \subset M \to B(0,3), \psi_i : V_i \subset N \to \mathbb{R}^n$ be atlas such that $F(W_i) \subset V_i$ and $\cup_i^{-1}(B(0,1)) = M$. Let $K_i$ be the closure of $\phi_i^{-1}(B(0,1))$ and $C(f) = \{x \in MDf_x$ is not surjective}. Then

$$f(C(f) \cap K_i) \subset \psi_i^{-1}(\psi_i \circ f \circ \phi_i^{-1})(C(\psi_i \circ f \circ \phi_i^{-1}))$$

is a closed set with empty interior, because, by sard’s theorem, $((\psi_i \circ f \circ \phi_i^{-1}))$ is countable union of closed subsets with empty interior. Thus the complement of is is residual.

**Theorem 1.7** If $F : M \times P \to N$ is a $C^\infty$ transformation which is transversal to a submanifold $S \subset N$, then the set of points $y \in P$ such that $F_y$ is transversal to $S$ is residual.

**Proof.** This is the immediate consequence of the previous two lemmas.

**Corollary 1.8** Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be of class $C^\infty$, $K \subset U$ is a compact set and $S \subset \mathbb{R}^n$ is a $C^\infty$ submanifold. Given $\epsilon > 0$ there exists a $C^\infty$ function $g : U \to \mathbb{R}^n$ such that

1. $g = f$ out of a compact neighborhood of $K$ contained in $U$.
2. $\|g - f\| < \epsilon$ in $U$.
3. $g$ is transversal to $S$ at points of $K$.

**Proof.** Let $\lambda : \mathbb{R}^m \to [0,1]$ such that $\lambda(x) = 1$ for $x \in K$. By theorem, the set of $v \in \mathbb{R}^n$ such that the map $x \in U \to f(x) + v$ is transversal to $S$ is residual and therefore dense. If $v \in \mathbb{R}^n$ belongs to this set and its norm is small enough then the map $g$ defined by $g(x) = f(x) + \lambda(x)v$ satisfies the three conditions.

**Corollary 1.9** Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be of class $C^\infty$, $K \subset U$ is a compact set. Let $S \subset J^1(U, \mathbb{R}^n) = U \times \mathbb{R}^m \times L(\mathbb{R}^m, \mathbb{R}^n)$ is a $C^\infty$ submanifold. Then given $\epsilon > 0$ there exists a $C^\infty$ function $g : U \to \mathbb{R}^n$ such that

1. $g = f$ out of a compact neighborhood of $K$ contained in $U$.
2. $\|g - f\|_{C^r} < \epsilon$ in $U$.
3. $j^1g$ is transversal to $S$ at points of $K$.

**Proof.** By theorem, the set $(v, A) \in \mathbb{R}^n \times L(\mathbb{R}^m, \mathbb{R}^n)$ such that if $g_{v,A}(x) = f(x) + v + A(x)$, then $j^1g_{v,A}$ is transversal to $S$, is a residual set. Let $\lambda : \mathbb{R}^m \to [0,1]$ as the previous proof. Then take $(v, A)$ as above with small enough norm, then $g(x) = f(x) + \lambda(x)(v + A(x))$ satisfies the conditions.

**Theorem 1.10** Let $S \subset N$ be a $C^\infty$ submanifold. $F \subset S$ is a closed set. Then define $\cap_F = \{f \in C^r(M, N) | \text{either } f(x) \notin F \text{ or } f \cap_x S\}$. $\cap_F$ is open and dense.
Proof. The openness has already been proved in theorem 1.4. As the $C^\infty$ transformations are dense it suffices to prove that each neighborhood $\mathcal{V}'$ of a $C^\infty$ transformation $f$ contains a transformation which is either transversal to $S$ or does not intersect $N$. Let

$$\varphi_i U_i \subset M \rightarrow \widetilde{U}_i \subset \mathbb{R}^m$$

$$\varphi_i V_i \subset M \rightarrow \widetilde{V}_i \subset \mathbb{R}^n$$

be local charts such that $f(U_i) \subset V_i$, $\{U_i\}$ are locally finite, $K_i$ is compact set contained in $U_i$ and $\cup_i K_i = M$. Let $\epsilon_i > 0$ small enough such that the neighborhood $\mathcal{V}'(f)$ consisting of functions $g$ such that $g(K_i) \subset V_i$ and $\|\psi_i \circ g \circ \varphi_i^{-1} - \psi_i \circ f \circ \varphi_i^{-1}\|_{C^r} < \epsilon_i$ at $\varphi_i(K_i)$ is contained in $\mathcal{V}$.

Define $A_i = \{g \in \mathcal{V}'(f), g$ is transversal to $S$ at $\varphi_i^{-1}(K_i)\}$ then by the corollary, $A_i$ is open and dense. Since the space $C^r(M,N)$ is Baire space, $\cap_i A_i$ is residual and thus dense in $\mathcal{V}'(f)$.

Corollary 1.11. $S \subset N$ is a closed submanifold, then $\{f \in C^r(M,N) | f \cap S\}$ is open and dense.

Corollary 1.12. $S \subset N$ is a submanifold, then $\{f \in C^r(M,N) | f \cap S\}$ is residual.

Example. Let $R_i \subset \mathbb{R}^3$ be line segments such that $\cup_i R_i$ is dense in $\mathbb{R}^3$. If $f : \mathbb{R} \rightarrow \mathbb{R}^3$ is any $C^1$ class curve, then there exists some curve $g$ which can be arbitrarily close to $f$, such that $g(\mathbb{R}) \cap R_i = \emptyset$ for all $i$.

Theorem 1.13 Let $S \subset J^k(M,N)$ is a submanifold of class $C^\infty$, not necessarily closed. Suppose $k \leq r - 1$ and consider the set $\{f \in C^r(M,N) | j^k f \cap S\}$. Then this set is residual (in particular, dense) and open if $S$ is closed.

Proof. Omitted.

Corollary 1.14 The set $Im^r(M,N) \subset C^r(M,N)$ with $r \geq 1$ is open and dense if $dim(N) \geq 2dim(M)$

Proof. The openness was already proved previously, it suffices to show the density of the immersions in $C^{r+1}(M,N)$. Let $P_k \subset J^1(M,N)$ be the set of points $(x,y,T) \in J^1(M,N)$, where $x \in M, y \in N$ and $T : TM_x \rightarrow TN_y$ is a linear map of rank $k$. Then $P_k$ is a submanifold of codimension $(m - k) \times (n - k)$ which is true if $n \geq 2m$ and $k < m$. By the previous theorem, the set of maps in $C^{r+1}(M,N)$ such that $j^1 f$ is transversal to each $P_k$ is residual (note that $P_k$ is not closed submanifold because it intersects $P_{k-1}$). Thus, if $f$ belongs this residual set, then $j^1 f(M) \cap P_k = \emptyset$ for $k < m$. Then $f$ is an immersion.

One application of this corollary is that, if $n = 1$, then in the space $C^r(M,\mathbb{R})$, the set $\{x \in M ; Df_x = 0\}$ is isolated, since it is a submanifold with codimension $m$.

Let $\mathcal{X}^r(M) \subset C^r(M, TM)$ be the vector space of vector fields. We say that $x \in M$ is a singularity of $X \in \mathcal{X}^r(M)$ if $X(x) = 0$. If $X : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a $C^r$ vector field, $r \geq 1$, a singularity $x \in X$ is a simple singularity if $DX_x$ has rank $n$, i.e., is an isomorphism.

Lemma 1.15 Let $r \geq 1$, $X : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a vector field of class $C^r$. Let $K \subset U$ be a compact set. Given $\epsilon > 0$ there exists a $C^r$ vector field $Y : U \rightarrow \mathbb{R}^m$ such that

1. $Y = X$ out of some compact neighborhood of $K$ which is contained in $U$.
2. $\|Y - X\|_{C^r} < \epsilon$ in $U$.
3. The singularities of $Y$ at $K$ are simple, i.e., $DY_x$ is an isomorphism if $x \in K$ and $Y(x) = 0$. 

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Proof. Omitted.

Theorem 1.16 Let \( \text{Im}(M, N) \subset C^r(M, N), r \geq 1 \) be the set of immersions, which is open and dense. Let

\[
\Delta_M = \{(x, y) \in M \times M | x = y\}
\]

\[
\Delta_N = \{(x, y) \in N \times N | x = y\}
\]

for each \( f \in \text{Im}(M, N) \) consider the map \( F_f : M \times M \setminus \Delta_M \to M \times N \), given by \( F_f(x, y) = (f(x), f(y)) \). The set of functions \( f \in \text{Im}(M, N) \) such that \( F_f \) is transversal to \( \Delta_N \) is residual and is also open if \( M \) is compact.

Proof. Since \( F_f : M \times M \setminus \Delta_M \) can be viewed as a manifold, and \( \Delta_N \) is a submanifold of \( N \times N \), one gets that the set of functions in a neighborhood of \( F_f \) in \( C^r(M \times M \setminus \Delta_M, N \times N) \) which is transversal to \( \Delta_N \) is residual. This implies that the set of \( f \in C^r(M, N) \) such that \( F_f \) is in this residual set is residual in \( C^r(M, N) \). If \( M \) is compact, we have that this set is open, too, by the Thom transversality theorem.

Corollary 1.17 If \( \dim N \geq 2 \dim M + 1 \) then the set of bijective immersions from \( M \) to \( N \) is residual and is also open if \( M \) is compact.

Proof. From previous results, we have that the set \( \text{Im}(M, N) \) is open in \( C^r(M, N) \). Then it suffices to prove that the set of 1–1 maps is dense in \( C^r(M, N) \). In the settings of the above theorem, this is equivalent with the requirement that \( F_f \) is transversal to \( \Delta_N \). In fact, since their dimensions satisfy \( \dim N \geq 2 \dim M + 1 \), the only way that \( F_f \) could be transversal to \( \Delta_N \) is that \( F_f(M \times M \setminus \Delta_M) \cap \Delta_N = \emptyset \), which is equivalent to \( f \) is bijective.

Corollary 1.18 (Whitney embedding) There exists an embedding of \( M \) to \( \mathbb{R}^{2m+1} \) where \( m = \dim M \).

Proof. As we have seen above, the set of proper maps is open and is not empty. Therefore they intersect the set of bijective immersions, which is dense. A proper bijective immersion is an embedding.