Differential Topology notes, 9

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1 Whitney Topology on $C^k(M,N)$

Let M and N be manifolds. If $U \subset J^r(M,N), r \geq 0$ is an open subset we define

$$\widehat{U} = \{f \in C^r(M,N); j^r f(M) \subset U\}$$

The family $\{\widehat{U} \subset C^r(M,N); U \subset J^r(M,N) \text{ is an open subset}\}$ forms a base of a topology in $C^r(M,N)$ which is called the Whitney topology.

Let

$$d_N : N \times N \to \mathbb{R}^+ \text{ and } d'^r : J^r(M,N) \times J^r(M,N) \to \mathbb{R}^+$$

be complete metrics, then for any $x \in M$, let $d^r(j^r f(x), j^r g(x)) = d'^r(j^r f(x), j^r g(x)) + d_N(f(x), g(x))$. Then if follows that

$$d_N(f(x), g(x)) \leq d^r(j^r f(x), j^r g(x))$$

Proposition 0. If $\varphi : M \to \mathbb{R}^+$ is a continuous and positive function define

$$\mathcal{V}(f;\varepsilon) = \{g \in C^r(M,N); d^r(j^r g(x), j^r f(x)) < \varepsilon(x), \forall x \in M\}$$

Then the family $\mathcal{V}(f;\varepsilon)$ is a neighborhood base of f in the Whitney topology.

Proof. The set $U_\varepsilon \subset J^r(M,N)$ defined by

$$j^r g(x) \in U_\varepsilon \Leftrightarrow d^r(j^r f(x), j^r g(x)) < \varepsilon(x)$$

is an open neighborhood of $j^r f(M)$ and $\mathcal{V}(f;\varepsilon)$ is the set of functions g such that $j^r g(M) \subset U_\varepsilon$. Thus, $\mathcal{V}(f;\varepsilon)$ is an open neighborhood of f. On the other hand, by the definition, given a neighborhood $\mathcal{V}$ of f in $C^r(M,N)$, there exists an open set $U \supset J^r f(M)$ such that $\widehat{U} \subset \mathcal{V}$. We will find a set of the form $\mathcal{V} \subset \widehat{U}$. Let $M = \bigcup K_i$ where K_i is compact and $K_i \subset Int(K_{i+1})$. Since $U \subset J^r(M,N)$ is an open set and $K_i \backslash Int(K_{i-1})$ is compact, there exists some $\varepsilon_i > 0$ such that if $d^r(j^r g(x), j^r f(x)) < \varepsilon_i$ then $j^r g(x) \in U$. As we have seen(using partition of unity), there exists a C^∞ function $\varepsilon : M \to \mathbb{R}^+$ such that $\varepsilon(x) < \varepsilon_i$ for any $x \in K_i \backslash Int(K_{i-1})$. Then $\mathcal{V}(f;\varepsilon) \subset \widehat{U} \subset \mathcal{V}$.

Proposition 1.
1. If M is compact then $C^r(M,N)$ is a complete metric space with countable base of open sets, i.e., there exists a countable dense subset.
2. If M is not compact then any $f \in C^r(M,N)$ does not admit a countable base of neighborhood. In particular, $C^r(M,N)$ is not metrizable.
3. If $f_n \in C^r(M,N)$ converge to f then there exists a compact subset $K \subset M$ and $n_0 \in \mathbb{N}$ such that $n \geq n_0$, $f_n(x) = f(x)$ for any $x \notin K$.

Proof.
1. If M is compact then

$$d(f,g) = \sup\{d^r(j^r f(x), j^r g(x)); x \in M\}$$

is finite and thus defines a metric in $C^r(M,N)$. The balls with center f form a neighborhood base of f in the Whitney topology. Let $f_n \in C^r(M,N)$ be a Cauchy sequence. Since $d_N(f_n(x), f_m(x)) < d^r(j^r f_n(x), j^r f_m(x))$ it follows that $\{f_n\}$ is a Cauchy sequence in N which is a complete metric space. So we can define $f_n(x) \to f(x)$ and the convergence is uniform. Thus, f is continuous. Let $\varphi_i : W_i \subset M \to B(0;3)$ and $\psi_i : V_i \subset N \to B(0;3)$ be local charts such that $f(W_i) \subset V_i$ and $M = \cup U_i$ where $U_i = \varphi_i^{-1}(B(0;1))$.

As M is compact we can find a finite number of charts satisfying the above properties. Since $f_n \to f$ uniformly, there exists n_0 such that if $n \geq n_0$, $f_n \circ \varphi_i^{-1}(B(0;2)) \subset V_i$. For each i consider the maps:

$$\psi_i \circ f_n \circ \varphi_i^{-1} : B(0;2) \to B(0;3)$$

$$D^j(\psi_i \circ f_n \circ \varphi_i^{-1}) : B(0;2) \to L_s^j(\mathbb{R}^m;\mathbb{R}^n)$$

Since f_n is a Cauchy sequence of metric d these sequences are Cauchy sequences. The first sequence converges to $\psi_i \circ f \circ \varphi_i^{-1}$ and the others converge to continuous functions. Thus $\psi_i \circ f \circ \varphi_i^{-1}$ is C^r and their derivatives until order r are the limits of the other sequences. It then follows that f_n converges to f for the metric d.

We want to find a countable base of open sets. Since $J^r(M,N)$ is a manifold, its topology has a countable base of open sets. The family W_i of finite unions of these U_i's is also a countable family of open set. We claim that the open sets $\widehat{W}_j = \{f \in C^r(M,N); j^r f(M) \subset W_j\}$ form a base of open sets. Let $\widehat{U} \subset C^r(M,N)$ be an open neighborhood of f such that $j^r f(M) \subset U$. Since $j^r f(M)$ as the image of a compact set, is compact we can find a cover of $j^r f(M)$ by finite number of U_i's, all contained in U. Their union is of the form W_j and $\widehat{W}_j \subset \widehat{U}$.

2. Let $f \in C^(M,N)$ and suppose for contradiction that there exists a countable base $\mathcal{V}_i, i = 1, \cdots$ of neighborhoods of f. Let $x_i \in M$ be a sequence going to infinity, i.e., for any compact subset $K \subset M$, there is some i_K such that if $i \geq i_K$, then $x_i \notin K$. If $\varepsilon_i > 0$ is small enough, there exists $f_i \in \mathcal{V}_i$ such that $d^r(j^r f(x_i), j^r f_i(x_i)) > \varepsilon_i > 0$.

Let $\varepsilon : M \to \mathbb{R}^+$ be a continuous function such that $\varepsilon(x) < \varepsilon_i$. Then, for each i, we have that $f \notin \mathcal{V} = \mathcal{V}(f,\varepsilon)$ Thus, $\mathcal{V}$ is not contained in any $\mathcal{V}_i$ and therefore the set $\{\mathcal{V}_i\}$ is not a neighborhood base.

3. Suppose that there exists no such compact set. Then there exists a sequence x_i going to infinity such that

$$d^r(j^r f(x_i), j^r f_{n_i}(x_i)) > \varepsilon_i > 0$$

Let $\varepsilon : M \to \mathbb{R}^+$ be positive function such that $\varepsilon(x_i) < \varepsilon_i$, then $f_{n_i} \notin \mathcal{V}(f;\varepsilon)$, a contradiction of the condition that f_n converges to f.