Definition 1. (Pullback of fiber bundles) Let $\pi : E \to M$ be a fiber bundle with respect to the cocycle $\delta_{ij} : U_i \cap U_j \to G$ and the action $\rho : G \to Diff^\infty(F)$. Let $f : N \to M$ be a $C^\infty$ transformation. Then $\{V_i = f^{-1}(U_i)\}$ is an open cover of $N$ and $\Phi_{ij} : V_i \cap V_j \to G$ defined by $\Phi_{ij} = \delta_{ij} \circ f$ is a cocycle on $N$. The fiber on $N$ with respect to this cocycle and the same action $\rho$ is denoted by $\pi^* : f^*(E) \to N$.

Then there exists a differentiable transformation $\hat{f} : f^*(E) \to E$ such that the diagram commute(omitted): $\pi \circ \hat{f} = f \circ \pi^*$, and the transformation $\hat{f}$ when restricted on each fiber of $F^*(E)$ is a diffeomorphism between onto the corresponding fiber of $E$. If $\pi : E \to M$ is a vector bundle then $\hat{f}$ is a linear isomorphism on each fiber.

Definition 2. (An Example of Universal bundles) Let the base manifold be the Grassmannian $G(k, n) = \{k\text{-planes in } \mathbb{R}^n\}$. Define the fiber bundle as $U(k, n) = \{(E, x) | E \in G(k, n), x \in E\}$. We can also make this a fiber bundle with structure group $GL(\mathbb{R}^k)$. Note also $U(k, n)$ can be regarded as a subspace of $G(k, n) \times \mathbb{R}^k$, since for each fixed $E \in G(k, n)$, $E$ is a $k$-subspace in $\mathbb{R}^n$.

Using the definition of pull back, if $f : M \to G(k, n)$ where $M$ is a manifold, then we have that $f^*(U(k, n))$ has fiber $\mathbb{R}^k$ and the fiber bundle $\pi' : f^*(U(k, n)) \to M$. Then, if we fix the dimension of $M$ as $k$, then letting $(n-k)k > 2k + 1$, we can embed $M$ into the space $G(k, n)$, then we have the following theorem,

Theorem 0.1 Any vector bundle on $M$ is isomorphic to $f^*(U(k, n))$ for some $n$.

Note that a vector bundle is just a fiber bundle where each bundle is a (real) vector space.

1 Tensor Fields

Let $V \subset \mathbb{R}^n$ be an open set. A $k$-form in $V$ is a $C^\infty$ map $\Phi : V \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ defined by $(x, (v_1, \cdots, v_k)) \mapsto \Phi(x; v_1, \cdots, v_k)$ which is linear in each of the variables $v_j$. If $U \subset \mathbb{R}^p$ is an open set and $\varphi : U \to V$ is a $C^\infty$ map, define the pullback of $\Phi$ under $\varphi$, such that

$$(\varphi^*\Phi)(x; v_1, \cdots, v_k) = \Phi(\varphi(x); (D\varphi_x)v_1, \cdots, (D\varphi_x)v_k)$$

We say that $\Phi$ is alternating if $\Phi(x; v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \text{sign}(\sigma)\Phi(x; v_1, \cdots, v_k)$ and symmetric if $\Phi(x; v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \Phi(x; v_1, \cdots, v_k)$ for any permutations $\sigma$ of the set $\{1, \cdots, k\}$. Clearly, $\varphi^*\Phi$ is symmetric (respectively, alternating) if $\Phi$ is symmetric (respectively, alternating).

A alternating $k$-forms is called a differential $k$-forms. A symmetric positive definite $k$-form is called a Riemannian metric. Note that if $\varphi$ is an immersion then the pullback of a Riemannian metric is also a Riemannian metric.

Note that the pull-back of a $k$-form does not require $\varphi$ to be diffeomorphism. We can also have $p \neq n$. On the other hand, the pull-back of vector field could be defined only when $\varphi$ is a diffeomorphism.

Denote by $(\mathbb{R}^n)^*$ the dual space of $\mathbb{R}^n$, i.e., the space of linear transformations of $\mathbb{R}^n \to \mathbb{R}$. Then, as a special case of the above $k$-forms, a vector field $X$ in $V \subset \mathbb{R}^n$ defines a $C^\infty$ function
\( T : V \times (\mathbb{R}^n)^* \to \mathbb{R} \), which is linear in the second variable, by \( T(x; \ell) = \ell(X(x)) \). Conversely, given any \( \Phi : V \times \mathbb{R}^n \to \mathbb{R} \), it can be viewed as a \( C^\infty \) vector field on \( V \) since for each \( x \in V \), the function \( \Phi(x, \cdot) \) is a linear functional from \( \mathbb{R}^n \) to \( \mathbb{R} \), which is an isomorphism between \( \mathbb{R}^n \) and itself.

A tensor bundle of type \((k, p)\) in an open set \( V \subset \mathbb{R}^n \) is a transformation

\[
T : V \times (\mathbb{R}^n)^* \times \cdots \times (\mathbb{R}^n)^* \times \mathbb{R}^n \cdots \times \mathbb{R}^n \to \mathbb{R}
\]

defined by \( T(x; \ell_1, \cdots, \ell_k, v_1, \cdots, v_p) \in \mathbb{R} \) which is linear in each of the variables \( \ell_i \in (\mathbb{R}^n)^* \) and \( v_j \in \mathbb{R}^n \).

The above vector field is a tensor field of type \((1, 0)\); while a \( k\)-form is a tensor field of type \((0, k)\).

**Pullback of tensor fields** Let \( \varphi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n \) be a diffeomorphism, then, the pullback of the tensor field \( T \) of type \((k, p)\) in \( V \) under \( \varphi \) is the tensor field \( \varphi^* T \) in \( U \), defined by

\[
(\varphi^* V)(x; \ell_1, \cdots, \ell_k, v_1, \cdots, v_p) = T(\varphi(x); \ell_1 \circ (D\varphi)^{-1}_x, \cdots, \ell_k \circ (D\varphi)^{-1}_x, v_1, \cdots, v_p)
\]

The map \( \varphi^* \) is a linear transformation and we have that

\[
(f \circ g)^* = (g^* \circ f^*)
\]

The set of tensor fields of type \((k, p)\) in an open set \( V \subset \mathbb{R} \) is a vector space \( T^k_p(V) \) and the transformation \( \varphi^* : T^k_p(U) \to T^k_p(V) \) is an isomorphism whose inverse is \((\varphi^{-1})^*\).

**Definition 1.1 (tensor fields on manifolds)** Let \( \{ \varphi_i : U_i \to \tilde{U}_i \subset \mathbb{R}^n, i \in I \} \) be an atlas in a manifold \( M \). A tensor field of type \((k, p)\) in \( M \) is a family \( T_i \) as tensor fields of type \((k, p)\) in \( \tilde{U}_i \), which is invariant under the change of coordinates,

\[
(\varphi_j \circ \varphi_i^{-1})^* (T_j|\varphi_j(U_i \cap U_j)) = T_i|\varphi_i(U_i \cap U_j)
\]

**Remark.** A \( C^\infty \) function \( f : M \to \mathbb{R} \) corresponds to a family of \( C^\infty \) functions \( f_i : \tilde{U}_i \to \mathbb{R} \) which are compatible under the change of coordinates:

\[
f_j \circ (\varphi_j \circ \varphi_i^{-1}) = f_i
\]

The derivatives of order \( p \) of each \( f_i \) is a symmetric \( p\)-form \( D^p f_i \) in each \( U_i \). But if \( p > 1 \) they can not define a tensor field of type \((0, p)\) in \( M \). This shows that we can not define derivatives of order greater than \( 1 \) between manifolds.

Let \( T(k, p) \) be the vector space of the multilinear function in \((\mathbb{R}^n)^* \times \cdots \times (\mathbb{R}^n)^* \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \). Consider the action \( GL(\mathbb{R}^n) \to GL(T(k, p)) \) defined by \( \rho(A)(T) = A^* T \), where

\[
A^* T(\ell_1, \cdots, \ell_k, v_1, \cdots, v_p) = T(\ell_1 \circ A^{-1}, \cdots, \ell_k \circ A^{-1}, Av_1, \cdots, Av_p)
\]

Then \( \rho \) is a representation of \( GL(\mathbb{R}^n) \) and therefore, together with the cocycle in \( M \) defined by the changes of coordinates, we can define a vector bundle \( T^k_p(M) \to M \) whose sections are the tensor fields of type \((k, p)\) in \( M \).

Now we focus on the special orthogonal group \( SO(m) \), consisting of all the orthogonal matrix with determinant \( 1 \). First recall the definition of a orientable manifold. We say a manifold \( M \) is orientable if and only if there exists an atlas \( \varphi_i : U_i \to \tilde{U}_i \subset \mathbb{R}^m \) such that for any \( x \in U_i \cap U_j \) we have \( \det(D(\varphi_j \circ \varphi_i^{-1})(x)) > 0 \), and \( \langle , \rangle_x \) is a Riemannian metric in \( M \).

Let the vector fields \( X^1_i : U_i \to TM, j = 1, \cdots, n \) be defined such that \( (D\varphi_i)_x(X^j_i(x)) = \frac{\partial}{\partial x^j} \) for any \( x \in U_i \), and thus \( X^1_i(x), \cdots, X^m_i(x) \) is a positive base of \( TM_x \). Using the Gram-Schmidt process, we can change these basis into orthonormal fields, namely, \( \{ Y^1_i, \cdots, Y^m_i \} \).
Define the map \( \gamma_{ij} : U_i \cap U_j \to SO(m) \) as \( \gamma_{ij}(x)(Y^i_1, \ldots, Y^i_m) = (Y^j_1, \ldots, Y^j_m) \) and this is a cocycle. With respect to this cocycle, we can consider the fiber bundle over a manifold \( M \) with fiber \( \mathbb{R}^m \), and then there exists tensor bundle such that at each fiber it is an inner product.

**Example. (direct sum and tensor product).** Let \( \pi_j : E_j \to M \) be vector bundles with respect to the cocycle \( \delta_{ij} : U_i \cap U_j \to G \) and the representation \( \rho_i : G \to GL(V_i) \) where \( V_i \) are vector spaces. Then, \( \rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2) \), \( \rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2) \) defined by

\[
(\rho_1 \oplus \rho_2)(g)(x \oplus y) = \rho_1(g)(x) \oplus \rho_2(g)(y)
\]

and

\[
(\rho_1 \otimes \rho_2)(g)(x \otimes y) = \rho_1(g)(x) \otimes \rho_2(g)(y)
\]

are representations of \( G \) and the corresponding vector bundles are denoted by \( E_1 \oplus E_2 \) and \( E_1 \otimes E_2 \).

The fibers at each point \( x \in M \) are isomorphic to \( \pi_1^{-1}(x) \oplus \pi_2^{-1}(x) \) and \( \pi_1^{-1}(x) \otimes \pi_2^{-1}(x) \).

## 2 The Jet Bundles

As we have already seen in the remark after definition 1.1, one can not define derivatives of order greater than 1 for functions between manifolds. On the other hand, however, we claim the following:

**Property.** Let \( f_i : U_i \to V_i, i = 1, 2 \) be two transformations of class \( C^k \) between open sets of Euclidean spaces and \( \varphi : U_1 \to U_2, \psi : V_1 \to V_2 \) be two \( C^k \) diffeomorphisms. Then, \( f_1 \) and \( f_2 \) have the same derivatives until order \( k \) at \( x \in U_1 \) if and only if \( \psi \circ f_1 \circ \varphi^{-1} \) and \( \psi \circ f_2 \circ \varphi^{-1} \) have the same derivatives until order \( k \). We use the convention that the 0 derivative is the function itself.

**The proof** of this property is left as exercise.

Let \( M \) and \( N \) be \( C^\infty \) manifolds and \( C^k(M, N) \) is the space of \( C^k \) maps between \( M \) and \( N \). If \( p \in M \) we define the following relation in \( C^k(M, N) \):

\[
f_1 \sim_p^k f_2 \text{ if and only if } f_1(p) = f_2(p) = q \text{ and } \psi \circ f_1 \circ \varphi^{-1} \text{ have the same derivatives until order } k \text{ at } \varphi(p) \text{ if } \psi \text{ and } \varphi \text{ are local charts around } q \text{ and } p.
\]

From the property above we know \( \sim_p^k \) is indeed an equivalence relation and each equivalence class of \( f \) is called the \( k \)-jet of \( f \) at \( p \) and is denoted as \( j^k f(p) \). The set

\[
J^k(M, N) = \{ j^k f(p); f \in C^k(M, N) \text{ and } p \in M \}
\]

is called the space of \( k \)-jets.

We also have the projection \( \pi : J^k(M, N) \to M \times N \), which associates for each \( k \)-jet \( j^k f(p) \) the pair \((p, q)\) where \( q = f(p) \) for some representative \( f \) in the equivalence class \( j^k f(p) \).

A \( C^r \) transformation \( f : M \to N \) with \( r \geq k \) induces a transformation \( j^k f : M \to J^k(M, N) \) which makes the diagram commute (diagram omitted): \( \pi_2 \circ \pi \circ j^k f = f \). Below we will show that \( \pi : J^k(M, N) \to M \times N \) has a structure of \( C^\infty \) bundle and that the function \( j^k f \) is of class \( C^{r-k} \).

For this, we will construct a cocycle in \( M \times N \) and an action of this cocycle on a manifold.

A natural candidate for the fiber is the space \( J^k(m, n) \) of \( k \)-jets of the \( C^k \) functions in \( \mathbb{R}^m \) to \( \mathbb{R}^n \) which takes \( 0 \) to \( 0 \).

We have a natural isomorphism

\[
J^k(m, n) \sim L([\mathbb{R}^m; \mathbb{R}^n]) \times L^2_s([\mathbb{R}^m; \mathbb{R}^n]) \times \cdots \times L^k_s([\mathbb{R}^m; \mathbb{R}^n])
\]
Let \( G^k(m) \subset J^k(m,m) \) be the open subset
\[
G^k(m) \sim GL(\mathbb{R}^m) \times L^2_+(\mathbb{R}^m;\mathbb{R}^n) \times \cdots \times L^k_+(\mathbb{R}^m;\mathbb{R}^n)
\]
It is clear that \( G^k(m) \) has a structure of Lie group defined by the Taylor’s polynomial of the composition of two Taylor’s polynomials, i.e., \( j^k f(0) \ast j^k g(0) = j^k (f \circ g(0)) \), where \( \ast \) is the product of the Lie group. So if \( g \in G^k(m) \), \( g \) is associated with the Taylor’s polynomial \( \varphi : \mathbb{R}^m \to \mathbb{R}^n \) which is an isomorphism locally at 0. The Taylor’s polynomial of the local inverse \( \varphi \) of class \( C \) is a fiber bundle with fiber \( J \) of the Lie group. So if \( g \), \( j \) is the composition of two Taylor’s polynomials, i.e.,
\[
\rho(g,h)(j^k f(0)) = j^k (h \circ f \circ g^{-1})(0).
\]
In order to get the structure of the bundle, it suffices to construct a cocycle of \( G^k(m) \times G^k(n) \) on \( J^k(m,n) \), as follows, given \( g \in G^k(m) \) and \( h \in G^k(n) \) and \( j^k f(0) \in J^k(m,n) \),
\[
\rho(g,h)(j^k f(0)) = j^k (h \circ f \circ g^{-1})(0).
\]
We leave it as an exercise to verify the cocycle’s condition. It follows that
\[
\gamma_{ij}^1(z) = \text{the taylor’s polynomial of the function } \hat{\alpha}_{ij} \circ \hat{\alpha}_{ij}^{-1}
\]
\[
\gamma_{ij}^2(z) = \text{the taylor’s polynomial of the function } \hat{\beta}_{ij} \circ \hat{\beta}_{ij}^{-1}
\]
We leave it as an exercise to verify the cocycle’s condition. It follows that \( \pi : J^k(M,N) \to M \times N \) is a fiber bundle with fiber \( J^k(m,n) \) and structure group \( G^k(m) \times G^k(n) \). The function \( j^k f \) is then of class \( C^{r-k} \) because in their coordinates, \( \hat{f} : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n \) is the expression of \( f \) in the trivialization of \( J^k(M,N) \) in \( \pi^{-1}(U \times V) \). The expression of \( j^k f \) is
\[
x \to (x, \hat{f}(x), D\hat{f}(x), D^2\hat{f}(x), \cdots , D^n\hat{f}(x))
\]
which is of class \( C^{r-k} \).