1 Fiber Bundles

**Definition 1.1** Tangent bundles of a manifold $M$, denoted by $TM$, is defined by

\[ TM = \{ (x,v) \in M \times TM_x \} \]

Let $\pi : TM \to M$ be the projection $\pi(x,v) = x$. We want to define a topology and a manifold structure in $TM$ such that $\pi$ is a $C^\infty$ submersion if $M$ is a $C^\infty$ manifold. Consider an atlas $\{(\varphi_i : U_i \subset M \to \mathbb{R}^m, i \in I)\}$ in $M$ and define the functions

\[ \Phi_i : \pi^{-1}(U_i) \subset TM \to U_i \times \mathbb{R}^m \]

by

\[ \Phi_i(x,v) = (x,(D_x\varphi_i)v) \]

Clearly $\Phi_i$ is a bijection and $\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^m \to (U_i \cap U_j) \times \mathbb{R}^m$ is the diffeomorphism $(x,v) \to (x,D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(x))v)$. Then, when $M$ is a $C^\infty$ differential manifold of dimension $m$, $TM$ is a $C^\infty$ differential manifold of dimension $2m$. The projection $\pi$ is a $C^\infty$ submersion, with $\pi^{-1}(p) = TM_p$. The set $\{U_i, \Phi_i\}$ is called a local trivialization of the fiber bundle.

**Exercise 1.** A $C^r$ vector field in $M$ is a $C^r$ map $X : M \to TM$ such that $\pi \circ X = Id_M$.

Since $\Phi_i$ are diffeomorphisms, define $\tilde{\Phi}_i : \pi^{-1}(U_i) \to \tilde{U}_i \times \mathbb{R}^m$ by $\tilde{\Phi}_i(x,v) = (\varphi_i(x),(D_x\varphi_i)(v))$. Then this is a $C^k$ atlas on $TM$. Define $\gamma_{ij} : U_i \cap U_j \to GL(\mathbb{R}^m)$ as the $C^\infty$ transformations $\gamma_{ij}(x) = (D(\varphi_j \circ \varphi_i^{-1})\varphi_i(x))$. By the chain rule, we have that, if $x \in U_i \cap U_j \cap U_k$ then

\[ \gamma_{ik}(x) = \gamma_{jk}(x) \cdot \gamma_{ij}(x), \]

where $\gamma_{jk}(x) \cdot \gamma_{ij}(x)$ is the composition of two linear transformations. The change of the local trivializations $\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^m \to (U_i \cap U_j) \times \mathbb{R}^m$ is written as

\[ \Phi_j \circ \Phi_i^{-1}(x,v) = (x,\gamma_{ij}(x)v) \]

where $\gamma_{ij}$ satisfies the above conditions.

**Definition 1.2** Let $\{U_i\}$ be an open cover of a manifold $M$. Let $G$ be a Lie group. A family of $C^\infty$ functions $\delta_{ij} : U_i \cap U_j \to G$ satisfying the condition $\delta_{ik}(x) = \delta_{jk}(x)\delta_{ij}(x)$ for any $x \in U_i \cap U_j \cap U_k$ is called a cocycle of $M$ taking values in the group $G$.

Recall that an action of the Lie group $G$ on a manifold $F$ is an homeomorphism $\rho : G \to Diff^\infty(F)$ such that the map $G \times F \to F$ defined by $(g,x) \mapsto \rho(g)(x)$ is of class $C^\infty$. If $F$ is a vector space, then $\rho(g)$ is an linear isomorphism for all $g \in G$. It is called a representation of the group $G$.  

**Definition 1.3** A fiber bundle with total space \( E \), base \( M \), fiber \( F \), projection \( \pi : E \to M \) and structure group \( G \), is a submersion \( \pi : E \to M \) to a \( C^\infty \) manifold \( M \), a \( C^\infty \) action \( \rho : G \to \text{Diff}(F) \) of the Lie group \( G \) on the fiber \( F \) and a cocycle \( \delta_{ij} : U_i \cap U_j \to G \) such that for each \( U_i \) there exists a diffeomorphism \( \Phi_i : \pi^{-1}(U_i) \to U_i \times F \) such that \( \pi \circ \Phi_i = \pi \), where \( \pi : U_i \times F \to U_i \) is the canonical projection to the first coordinate; and such that \( \Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \) is such that \( \Phi_j \circ \Phi_i^{-1}(x, y) = (x, \rho_{ij}(x)(y)) \), where \( \rho_{ij} = \rho \circ \delta_{ij} \).

The functions \( \rho_{ij} \) are called the transition functions. If \( \rho \) is a representation of \( G \) then each fiber \( \pi^{-1}(x) \) has a vector space structure such that the restriction of \( \Phi_i \) on \( \pi^{-1}(x) \) is an isomorphism between \( \pi^{-1}(x) \) and \( F \). We say that \( \pi \) is a vector fiber bundle. An example is the tangent bundle of \( M \).

**Theorem 1.4** Given any cocycle \( \delta_{ij} : U_i \cap U_j \to G \) on a manifold \( M \). Let \( \rho \) be a \( C^\infty \) action. Then there exists a fiber bundle \( \pi : E \to M \) with fiber \( F \) and group structure \( G \), the transition function being \( \rho_{ij} = \rho \circ \delta_{ij} \).

**Proof.** Let \( \hat{E} \) be a disjoint union \( \bigsqcup_i (U_i \times F) \) and \( \hat{\pi} : \hat{E} \to M \) is the relation defined by

\[
(x, v) \sim (y, w) \iff \begin{cases} x = y, & \text{and} \\ w = \delta_{ij}(x)v, & \text{if } x \in U_i \cap U_j. \end{cases}
\]

Using the definition of cocycles, it can be verified the above is an equivalence relation. Let \( E \) be the set of equivalence classes and \( q : \hat{E} \to E \) be the quotient map. Equip \( E \) with the quotient topology we have that there exists a unique continuous map \( \pi : E \to M \) such that \( \hat{\pi} = \pi \circ q \). For each \( i \), the map \( \Phi_i : U_i \times F \to \pi^{-1}(U_i) \subset E \) defined by the composition the inclusion \( i : U_i \times F \hookrightarrow \hat{E} \) and the projection map \( q \) is a homeomorphism. By the definition of equivalence relation it follows that the homeomorphism

\[
\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F
\]

is given by

\[
(x, v) \mapsto (x, \delta_{ij}(x)(v))
\]

Thus there exists a unique manifold structure in \( E \) satisfying the conditions of the theorem.

**Definition 1.5** A \( C^r \) section of a fiber bundle \( \pi : E \to M \) is a \( C^r \) map \( X : M \to E \) such that \( \pi \circ X = \text{Id}_M \).

**Proposition.** Let \( \pi : E \to M \) be a fiber bundle with group structure \( G \) and fiber \( F \), cocycle \( \delta_{ij} : U_i \cap U_j \to G \) and the action \( \rho : G \to \text{Diff}(F) \). We can identify a section \( X \) of \( \pi \) with a family \( X_i : U_i \to F \) of \( C^k \) functions satisfying the compatibility condition of as follows:

\[
x \in U_i \cap U_j \Rightarrow X_j(x) = \rho_{ij}X_i(x)
\]

**Example 0.** Let \( \delta_{ij} : U_i \cap U_j \to GL(\mathbb{R}^m) \) be defined for an atlas in \( M \). Let \( \rho : GL(\mathbb{R}^m) \to \text{Diff}(\mathbb{R}^m) \) given by \( \rho(A)(x) = Ax \). Then, the resulting fiber bundle is the tangent bundle of \( M \) and the \( C^k \) sections are the vector fields in \( M \).

**Example 1.** Observe the action \( GL(\mathbb{R}^m) \times (\mathbb{R}^m)^* \to (\mathbb{R}^m)^* : (A, \lambda) \mapsto \lambda \circ A^{-1} \). Similarly, let \( A \in GL(\mathbb{R}^m) \), and let \( L^k(\mathbb{R}^m, \mathbb{R}) \) be the space of all the \( k \)-linear transformations, which is a vector space. Define \( A^* : L^k(\mathbb{R}^m, \mathbb{R}) \to L^k(\mathbb{R}^m, \mathbb{R}) \) by \( (A^* L)(v_1, \ldots, v_k) = L(A^{-1}v_1, \ldots, A^{-1}v_k) \). Then we have \( (A \circ B)^* = B^* A^* \). Let \( \rho : GL(\mathbb{R}^m) \times L^k(\mathbb{R}^m, \mathbb{R}) \to L^k(\mathbb{R}^m, \mathbb{R}) \) be defined as \( \rho(A, L) = A^* L \). Any section of this fiber bundle is actually a \( k \)-differential form in \( M \).
A special case is when \( k = 2 \). We use \( S_2(TM) \rightarrow M \) to denote this vector bundle, where each \( \pi^{-1}(x) \approx S_2(\mathbb{R}^m) \) is the space of all the symmetric bilinear maps \( TM_x \times TM_x \rightarrow \mathbb{R} \). Note that the space of all the positive definite bilinear transformations, denoted by \( C_+(\mathbb{R}^m) \), is a cone (i.e., for any \( \lambda \in \mathbb{R}^+ \) and any \( L \in S_2(\mathbb{R}^m) \)) in the vector space of all the symmetric bilinear transformation. So we have that the set \( C_+(TM) \) is an open subset of \( S_2(TM) \). A \( C^\infty \) section of \( S_2(TM) \) which takes values in \( C_+(TM) \) is a Riemannian metric on \( M \).

A simpler case is when \( k = 1 \). \( L(\mathbb{R}^m, \mathbb{R}) = (\mathbb{R}^m)^* \), while the corresponding bundle is the cotangent bundle.

If in the fiber bundle, the fiber is the Lie group \( G \), and the action is \( G \) at the same time. We still have several different interesting structures. For example, \( \varphi_1 : G \times G \rightarrow G, \varphi(g, h) = g \cdot h \); while \( \varphi_2 : G \times G \rightarrow G, \varphi(g, h) = g \cdot h \cdot g^{-1} \) can define different bundles.

**Definition 1.6** Let \( \delta_{ij} : U_i \cap U_j \rightarrow G \) be a cocycle in a manifold \( M \). The fiber bundle in \( M \) with respect to the action \( \rho : G \rightarrow Diff(G) \) given by \( \rho(g)(h) = gh \) is called a principal bundle.

**Definition 1.7** A right action of a group \( G \) on a manifold \( E \) is an anti-homomorphism \( \gamma : G \rightarrow Diff(E) \) such that \( \gamma(g_1, g_2) = \gamma(g_2)\gamma(g_1) \) and that the map \( E \times G \rightarrow E : (x, g) \rightarrow \gamma(g)(x) \) is \( C^\infty \).

**Theorem 1.8** A locally trivial fiber bundle \( \pi : E \rightarrow M \) is a principal bundle with respect to some cocycle \( \delta_{ij} : U_i \cap U_j \rightarrow G \), then there exists a right action of \( G \) on \( E \), \( E \times G \rightarrow E \) which preserves the fibers and acts transitively and has no fixed points at each fiber.

**Proof.** Let \( \pi : E \rightarrow M \) be a principal bundle with respect to the cocycle \( \delta_{ij} \) and let \( \Phi_i : \pi^{-1}(U_i) \rightarrow U_i \times G \). Since \( \Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times G \) is defined by \( (x, g) \rightarrow (x, \delta_{ij}(x)g) \) we have that \( \Phi_j \circ \Phi_i^{-1} \) is compatible with the right action of \( G \) on \( (U_i \cap U_j) \times G \) given by \( (x, g)h = (x, gh) \). Then the right action of \( G \) on \( \pi^{-1}(U_i \cap U_j) \) induced by \( \Phi_i \) coincides with the induced action from \( \Phi_j \) and, therefore, we have got a right action of \( G \) on \( E \) which satisfies the requirements of this theorem.

**Example.** A frame bundle on \( M \) is a principal bundle with group structure \( GL(\mathbb{R}^m) \). The fiber \( F \) associated to this bundle is the set of all the frames with respect to a finite dimensional vector space, which is homeomorphic to the space \( GL(\mathbb{R}^m) \). The right group \( GL(\mathbb{R}^m) \) action on itself is of course defined by \( \rho(A)B = BA \).