1 Density of functions in the class $C^\infty$

Recall that $f : M \to \mathbb{R}^k$ is a $C^r$ function if and only if there exists some $C^r$ atlas $\varphi_i : U_i \subset M \to \mathbb{R}^k$ such that $f \circ \varphi_i^{-1} : U_i \to \mathbb{R}^k$ is of class $C^r$. The space of $C^r$ functions is a vector space of infinite dimension.

Let $M$ be a $C^\infty$ manifold, we will define a topology in the space $C^0(M, \mathbb{R}^k)$ called the Whitney $C^0$ topology and show that the functions of class $C^\infty$ are dense in this topology. For each open subset $U \subset M \times \mathbb{R}^k$ let $\tilde{U} \subset C^0(M, \mathbb{R}^k)$ be the set of functions $g$, whose graphs $\{(x, g(x)) \in M \times \mathbb{R}^k : x \in M\}$ is contained in $U$. It is not difficult to verify the family $\tilde{U}_i$ defines a topology in $C^0(M, \mathbb{R}^k)$. We want to construct a neighborhood basis of each function $f \in C^0(M, \mathbb{R}^k)$.

Let $K_i$ be a countable family of compact sets covering $M$ such that $K_i$ is contained in the interior of $K_{i+1}$ (cf. notes 4). Consider the the compact subsets $L_i = K_i \setminus \text{Int}(K_{i-1})$, which are compact sets, too. Let $\tau = (\epsilon_i)_{i=1}^\infty$ be a sequence of positive numbers and then define

$$V(f, \tau) = \{g \in C^0(M, \mathbb{R}^k) : \|f(x) - g(x)\| \leq \epsilon_i \text{ for any } x \in L_i\}.$$ 

We claim this is a neighborhood system of the function $f$ in $C^0(M, \mathbb{R}^\infty)$. Since $L_i$ is compact, the set $U = \{(x, y) \in M \times \mathbb{R}^k : \|y - f(x)\| \leq \epsilon_i \text{ if } x \in L_i\}$ is open. Thus, $V(f, \tau) = \tilde{U}$ is an open neighborhood of $f$. On the other hand, if $V$ is an open subset of $M \times \mathbb{R}^k$ which contains the graph of $f$, then since $L_i$ is compact, it follows that there exists $\epsilon_i > 0$ such that if $x \in L_i$ and $\|y - f(x)\| \leq \epsilon_i$, then $(x, y) \in V$. Thus, taking $\tau = (\epsilon_i)$ we have $V(f; \epsilon) \subset V$, so we have obtained the family $V(f; \epsilon)$ is a neighborhood system of $f$.

Moreover, for each given $(\epsilon_i)$, we can find a $C^\infty$ function $\epsilon : M \to \mathbb{R}^+$, such that $\epsilon(x) < \epsilon_i$ for any $x \in L_i$. It follows that the family $V(f, \epsilon) = \{g \in C^0(M, \mathbb{R}^k) : \|f(x) - g(x)\| \leq \epsilon(x)\}$ is also a neighborhood system.

**Theorem 1.1** The set of all the $C^\infty$ functions is dense in $C^0(M, \mathbb{R}^k)$

**Proof.** Let $V(f; \epsilon)$ be a neighborhood of $f$. We want to show that this neighborhood contains a function of class $C^\infty$. As in the previous notes, we consider a Riemannian metric in $M$ such that the length of a curve connecting a point of $K_i$ to a point in the complement of $K_{i+1}$ is greater than or equal to 1, and let $d$ be the corresponding metric. As the restriction of $f$ on a compact set is uniformly continuous, we have that for each $i$ there exists $0 < \delta_i < 1$ such that if $x \in L_i$ and $d(y, x) < \delta_i$, then $\|f(y) - f(x)\| < \min\{\epsilon_{i-1}, \epsilon_i, \epsilon_{i+1}\}$.

Observe that if $x \in L_i$, then the ball $B(x; \delta_i)$ is contained in the compact set $K_{i+1} \setminus \text{Int}(K_{i-2})$. For each $i$ we can take a finite cover of $L_i$ using balls centered at one point in $L_i$ and with radius $\delta_i$. The collection $A = \{B_i\}$ of all such balls is a locally finite cover of the space $M$.

Then there exists a partition of unity $\lambda_i$ subordinate to the cover $A$. Denote these balls as $B_j = B(x_j; r_j)$ with centers $x_j$. Then define $g(x) = \sum_{n=1}^\infty \lambda_n(x)f(x_n)$. Since for each $x \in M$ there is some neighborhood $V_x$ where only finitely number of $\lambda_i$’s do not vanish, $g$ is well defined and is actually of class $C^\infty$. 

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Let \( x \in L_i \), if \( \lambda_j(x) \neq 0 \), then \( x \) is contained in a ball centered at \( x_j \in L_{i-1} \cup L_i \cup L_{i+1} \), and therefore, \( \| f(x) - f(x_j) \| < \epsilon_i \).

At last, we arrive at the estimation:

\[
\| f(x) - g(x) \| \\
\leq \| \sum_n \lambda_n(x) f(x) - \sum_n \lambda_n(x) f(x_n) \| \\
\leq \sum_n \lambda_n(x) \epsilon = \epsilon,
\]

which completes the proof.

**Corollary 1.2** Let \( f \in C^0(M, \mathbb{R}^k) \) be such that the restriction of \( f \) on an open subset \( V \subset M \) is of class \( C^\infty \). Let \( K \subset V \) be a compact subset. Given a neighborhood \( V \) of \( f \) there exists a function \( g \) of class \( C^\infty \) such that \( g \in V \) and \( g(x) = f(x) \) for any \( x \in K \).

**Proof.** Let \( \phi : M \to [0, 1] \) be a function of class \( C^\infty \) which takes value 1 in \( K \) and 0 out of a neighborhood \( U \) of \( K \), whose closure \( \overline{U} \) is contained in \( V \). Then for the fixed neighborhood \( V \) of \( f \), we can choose a \( C^\infty \) function \( g \in V \). Then we find that the function \( \phi(x)f(x) + (1 - \phi(x))g(x) \in V \), which is the desired function.

**Theorem 1.3** Let \( D^n \) be a closed ball with radius 1 in \( \mathbb{R}^n \) and \( S^{n-1} \) is the unit sphere. Show that, there does not exist a retraction of \( D^n \) to the sphere \( S^{n-1} \), i.e., a continuous function \( f : D^n \to S^{n-1} \) such that \( r(x) = x \) for any \( x \in S^{n-1} \)

**Proof.** Suppose for contradiction that there exists a retraction \( f : D^n \to S^{n-1} \). Let \( f : \mathbb{R}^n \to S^{n-1} \) defined by

\[
f(x) = \begin{cases} 
  r(2x), & \text{if } \| x \| \leq \frac{1}{2}, \\
  \frac{x}{\| x \|}, & \text{if } \| x \| \geq \frac{1}{2}
\end{cases}
\]

Then \( f \) is a continuous function and when restricted on the complement of the ball with radius \( \frac{1}{2} \) it is a \( C^\infty \) function. By the previous Corollary, we can find a \( C^\infty \) function \( g \) which coincides with \( f \) at a neighborhood of the sphere \( S^{n-1} \), such that \( \| f(x) - g(x) \| \leq \frac{1}{2} \) if \( \| x \| \leq 1 \). As \( \| f(x) \| = 1 \) for any \( x \in D^n \) we have that \( g(x) \neq 0 \) and therefore \( \rho(x) = \frac{g(x)}{\| g(x) \|} \) is a \( C^\infty \) retraction which indeed coincides with \( f \) in a neighborhood of this sphere.

By Sard’s theorem, we can therefore take a regular value \( y \) of \( \rho \). Since each connected component of the pre-image of \( y \) is a manifold of dimension 1, the connected component containing \( y \), \( V \), is a closed interval. So \( V \cap D^n \) is an closed interval, too, having \( y \) as one of its endpoints. The other endpoint, however, must lie in the sphere \( S^{n-1} \), which is a contradiction.

**Corollary 1.4** (Brouwer fixed point theorem) Each continuous function \( f : D^n \to D^n \) has a fixed point.

**Proof.** If \( f(x) \neq x \) for all \( x \in D^n \) we can define a continuous retraction \( r : D^n \to S^{n-1} \) by: \( r(x) \) is the intersection of sphere \( S^{n-1} \) and the ray starting from \( f(x) \) going through \( x \).

At the last paragraph of this section, we give a property which generalized Liouville’s theorem, which states that bounded entire functions must be constant.

**Property.** \( M \) is a complex manifold, compact and connected. \( f : M \to \mathbb{C} \) is a holomorphic function, then \( f \) must be constant.

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Proof. Let \( \phi : U_i \subset \widetilde{U}_i \to \mathbb{C} \) be an atlas. Since the space is compact, there is a finite cover with charts. Suppose \( |f| \) achieve its maximum value at some point \( z_0 \) and \( z_0 \in U_j \). By the maximum modulus principle, we have that \( f \) must be constant within \( U_j \). Then using induction and maximum modulus principle for many more times, we get our conclusion.

2 manifolds with boundary; collar neighborhood

Set \( \mathbb{H}^n = \{ x \in \mathbb{R}^n ; x^n \geq 0 \} \) and call it a super halfspace. (n-halfplane)

Definition 2.1 \( f : U \to \mathbb{H}^n \to \mathbb{H}^p \subset \mathbb{R}^p \). We say \( f \) is \( C^n \) if for any \( x \in U \) there exists some neighborhood \( V_x \subset \mathbb{R}^n \) and a \( C^n \) function \( f_x : V \to \mathbb{R}^p \) such that \( f_x = f \) in \( V \cap \mathbb{H}^n \).

Definition 2.2 \( M \) is called a manifold with boundary if there exists a family of homeomorphisms \( \varphi_i : U_i \subset M \to \widetilde{U}_i \subset \mathbb{H}^m \) such that all \( \varphi_j \circ \varphi_i^{-1} \) are diffeomorphisms and \( x \in \partial M \) if and only if there is local chart \( \varphi_i : U_i \subset M \to \widetilde{U}_i \subset \mathbb{H}^m \) such that \( \varphi_i(x) \in \partial \mathbb{H}^m \).

Note in this case, the boundary \( \partial M \) is a manifold of dimension \( m - 1 \).

With the notion of manifolds with boundaries, we can define the vector fields on the manifolds with boundaries: A vector field in a manifold with boundary is the restriction of a vector field in \( \mathbb{R}^n \) to the halfspace \( \mathbb{H}^m \). In an vector space \( V \) of finite dimension \( n \), we say that two bases are equivalent with each other if the transition matrix of these bases has positive determinant. Clearly, there are exactly two equivalent classes. An orientation of the vector space is a choice of one of these two classes. We can call one base a positive base and another one which is not equivalent is called a negative base.

An orientation of a manifold \( M \) is a choice of an orientation for each tangent space such that for each local chart \( \varphi : U \subset M \to \widetilde{U} \) with \( U \) connected, we have that the derivative of \( \varphi \) at the point \( x \), \( D\varphi : TM_x \to \mathbb{R}^n \), either preserves the orientation at each point \( x \), or inverts the orientation at each point \( x \). Such manifold, on the other hand, is called orientable.

If \( M \) is such an orientated manifold we can choose an atlas \( \varphi_i : U_i \to \widetilde{U}_i \subset \mathbb{H}^n \) such that the derivative of the changes of coordinates are isomorphisms of \( \mathbb{R}^n \) which preserve the orientation.

If \( M \) is an orientable manifold with boundary, then \( \partial M \) is also an orientable manifold. Consider the orientation of \( \partial M \) such that a base \( [v_1, \ldots , v_{n-1}] \) of \( T(\partial M)_x \) is positive if \( [v_1, \ldots , v_{n-1}, v] \) is a positive base of \( TM_x \), where \( v \in TM_x \) is a vector which is transversal to the subspace \( T(\partial M)_x \) and that pointed to the interior of \( M \), i.e., for a chart \( \varphi : I \subset M \to \widetilde{U} \subset \mathbb{H}^n \) we have that \( D\varphi_x(v) \in \mathbb{H}^n \subset \mathbb{R}^n \).

Theorem 2.3 (collar neighborhood of boundary) Let \( M \) be a \( C^\infty \) manifold with boundary whose boundary \( \partial M \) is compact. Then there exists a neighborhood \( V \) of \( \partial M \) and a class \( C^\infty \) diffeomorphism, \( \phi : \partial M \times [0,1) \to V \) such that \( \phi((x,0)) = x \) for any \( x \in \partial M \).

Proof. Consider a finite cover \( U_i \) of the boundary \( \partial M \) by open subsets of \( M \) such that there exists local charts \( \varphi_i : W_i \to \widetilde{W}_i \subset \mathbb{H}^n \) satisfying \( U_i \subset \widetilde{U}_i \subset V_i \subset \overline{V}_i \subset W_i \) with \( \overline{U}_i \) and \( \overline{V}_i \) compact sets. Let \( \lambda_i : \mathbb{H}^n \to [0,1] \) be a \( C^\infty \) function which takes value 1 in \( U_i \) and takes 0 out of \( V_i = \varphi_i^{-1}(V) \). Let \( \overline{X}_i \) be the vector field obtained by multiplying the filed \( \frac{\partial}{\partial \lambda_i} \) in the space \( \mathbb{H}^n \) by the function \( \lambda_i \).

Define now \( X_i = \varphi_i^*(\overline{X}_i) \). Then \( X_i \) is a vector field of class \( C^\infty \), which vanished out of a compact set and such that for each point \( x \in \partial M \), either \( X_i(x) = 0 \) or \( X_i(x) \) is transversal to \( \partial M \) and pointing to the interior of \( M \). Now define \( X = \sum_i X_i \), we then have that \( X \) is a \( C^\infty \) vector field.
which vanishes out a compact neighborhood of \( \partial M \) and for any point \( x \in \partial M \), \( X(x) \) is transversal to \( \partial M \) pointing to the interior of \( M \).

Therefore, there exists some \( \epsilon > 0 \) and a \( C^\infty \) map \( \psi : \partial M \times [0, \epsilon) \to M \) such that \( \psi(x, 0) = x \) and \( t \to \psi(x, t) \) is an integral curve of \( X \). Take \( \epsilon \) small enough we can suppose that \( \psi \) is a diffeomorphism onto a neighborhood \( V \) of \( \partial M \) in \( M \). At last, define \( \Phi(x, t) = \psi(x, et) \).

Using the above theorem, we can construct manifold with boundary with a pair of manifolds with boundaries. We state this as our final problem in this notes.

**Theorem 2.4** Let \( M \) and \( N \) be manifolds of class \( C^\infty \) with compact boundaries. Let \( f : \partial M \to \partial N \) be a diffeomorphism of class \( C^\infty \). Then there exists \( M \cup_f N \), which is a manifold with boundary, a submanifold \( S \subset M \cup_f N \) and embeddings \( i_f : M \to M \cup_f N \) and \( j_f : N \to M \cup_f N \) such that:

- \( i_f(M) \cup j_f(N) = M \cup_f N \).
- \( i_f(\partial M) \cap j_f(\partial N) = \emptyset \).
- \( i_f|_{\partial M} : \partial M \to S \) and \( j_f|_{\partial N} : \partial N \to S \) are diffeomorphism such that \( i_f = j_f \circ f \).

**Proof.** In the disjoint union of \( M \) and \( N \), \( M \sqcup N \), consider the following equivalence relation:

\[
x \sim y \Leftrightarrow \begin{cases} 
  \text{or } x = y; \\
  \text{or } x \in \partial M \text{ and } y = f(x); \\
  \text{or } x \in \partial N \text{ and } y = f^{-1}(x).
\end{cases}
\]

Let \( M \cup_f N \) be the quotient space with the quotient topology and \( q : M \sqcup N \to M \cup_f N \) be the quotient map. Let \( \Phi_M : \partial M \times [0, 1) \to V_M \subset M \) and \( \Phi_N : \partial N \times [0, 1) \to V_N \subset M \) be the collar neighborhoods of the boundaries \( \partial M \) and \( \partial N \). Observe that \( V = q(V_M \cup V_N) \) is a neighborhood of \( S = q(\partial M) = q(\partial N) \), then we define \( \Psi : \partial M \times (-1, 1) \to V \subset M \cup_f N \) as

\[
\Psi(t, x) = \begin{cases} 
  \Psi_M(-t, x) \text{ if } t \leq 0; \\
  \Psi_N(t, f(x)) \text{ if } t \geq 0.
\end{cases}
\]

Then \( \Psi \) is a homeomorphism onto \( V \). Let \( i_f \) be the composition of \( q \) and the inclusion of \( M \) into \( M \sqcup N \) and \( j_f \) be the composition of \( q \) and the inclusion of \( N \) into \( M \sqcup N \). Then there exists a unique structure of the manifold in \( M \cup_f N \) such that \( i_f, j_f \) and \( \Psi \) are all \( C^\infty \) embeddings.

**Example 1.** Let \( M \) and \( N \) be two manifolds of the same dimension, say, \( n \) (without boundaries). Then let \( D_M \) be an open subset of \( M \) which is homeomorphic to the unity disc in \( \mathbb{R}^n \). Similarly, Then let \( D_N \) be an open subset of \( N \) which is homeomorphic to the unity disc in \( \mathbb{R}^n \) Then from the above theorem, we can construct a new manifold \( (M - D_M) \cup (N - D_N) \) with the corresponding differential structure.

**Exercise.** Let \( M = M = D^2 \times S^1 \), where \( D^2 \) is the disc in \( \mathbb{R}^2 \) and \( S^1 \) is the circle. Obviously, \( \partial M = \partial N = S^1 \times S^1 \). We then define

\[
f : S^1 \times S^1 \to S^1 \times S^1, f(x, y) = (x, y),
\]

\[
g : S^1 \times S^1 \to S^1 \times S^1, g(x, y) = (y, x).
\]

Show that \( M \cup_f N \) is diffeomorphic to \( S^1 \times S^2 \) while \( M \cup_f N \) is diffeomorphic to \( S^3 \). In general, actually, we have \( \partial(D^{n-1} \times S^1) = S^{n-2} \times S^1 = \partial(S^{n-2} \times D^2) \).