1 Tubular flow theorem

Let $M$ be a $C^\infty$ manifold, $X$ be a complete $C^k$ vector field with $k \geq 1$, then the map from $t$ to $X_t$ is a homomorphism from $\mathbb{R}$ to $Diff^k(M)$, the group of diffeomorphisms from $M$ to $M$.

**Proposition 1.** For each $x, y \in M$ there exists $C^\infty$ diffeomorphism $f : M \to M$ such that $f(x) = y$.

**Proposition 2.** Let $\gamma : [0, 1] \to M$ be an embedding. There exists a neighborhood $V \supset \gamma([0, 1])$ and a $C^\infty$ diffeomorphism $\varphi : V \to B(0, 1) \subset \mathbb{R}^n$, where $B(0, 1)$ is an open ball.

**Lemma 1.1.** Suppose $M$ is connected, then for any $x, y \in M$, there exists an curve embedding $\gamma : [0, 1] \to M$ such that $\gamma(0) = x, \gamma(1) = y$.

**Proof.** Fix the point $x$ and let

$$A = \{ z \in M : \text{there exists an curve embedding in } M \text{ which passes both } x \text{ and } z \}$$

If $z_0 \in A$ and suppose that $\gamma : [a, b] \to M$ satisfy $\gamma(a) = z, \gamma(b) = z_0$. Suppose also $V$ is a neighborhood of $z_0$ which is diffeomorphic under $\varphi$ to an open ball, $B(0, R) \in \mathbb{R}^m$, with $\varphi(z_0) = 0$, we can suppose there is some $t_0 \in (a, b)$ such that $\gamma(t_0) \in V$.

Now we consider in the ball $B(0, R)$ a smaller ball $B(0, r)$ such that $\varphi(\gamma(t_0)) \notin B(0, r)$. Now take any $v \in B(0, r)$, Define $\alpha : [t_0, b] \to B(0, R)$ by $\alpha(t) = \varphi(\gamma(t)) + e^{t - t_0} v$. Let

$$\alpha'(t) = \left\{ \begin{array}{ll} \gamma(t), & \text{if } t \in [a, t_0]; \\ \varphi^{-1}(\alpha(t)), & \text{if } t \in [t_0, b]. \end{array} \right.$$ 

Then $\alpha$ links $x$ and $z = \varphi^{-1}(v)$. Since $\varphi^{-1}(v)$ is an open neighborhood of $z_0$, we have indeed proved that $A$ is an open set.

For any $z_0 \in A$, the same arguments as above implies that we can construct a $C^\infty$ curve from $x$ to $z_0$, showing that $A$ is also closed. Thus, $A = M$.

Before stating the next theorem, we introduce the notation of $\frac{\partial}{\partial t}$, which, in the Cartesian product space $\mathbb{R} \times S$, denote the vector field $X$ that at each point $(s, x)$, $X(s, x) = (1, 0)$, i.e., it is tangent to the curve $\gamma(t) = (s + t, x)$ at the moment $t = 0$.

Note that the theorem of local flow, stated without proof in the previous notes, is also useful in the proof of the tubular flow theorem:

**Theorem 1.2 (Tubular Flow Theorem)** Let $\gamma : [0, \ell] \to M$ be an integral curve embedded into the vector field of class $C^k$, i.e., $\gamma'(t) \neq 0$ for any $t$ and $\gamma$ is injective. Let $B \subset \mathbb{R}^{m-1}$ be a unit ball, then there exists a diffeomorphism of class $C^k$, $\Psi : (-\varepsilon, \ell + \varepsilon) \times B \to W$, such that $\Psi^* X = \frac{\partial}{\partial t}$ and $W$ is a neighborhood of $\gamma([0, \ell])$.

**Proof.** Suppose the manifold $M$ is of dimension $m$.
First, by the properties from the previous notes, the integral curve can be extended to a larger integral, say, $(-\varepsilon, \ell + \varepsilon)$, then fix some $t_0 \in [0, \ell] \subset (-\varepsilon, \ell + \varepsilon)$ and observe that at the point $\gamma(t_0)$ we have that $X(\gamma(t_0)) = \gamma'(t_0) \neq 0$.

Now there exists a neighborhood $V$ of $\gamma(t_0)$ and a local coordinate $\varphi$ such that $\varphi(V) \subset \mathbb{R}^n$ and $\varphi(\gamma(t_0)) = 0$, we can also choose a proper basis of the space $\mathbb{R}^n$ such that $\varphi \cdot X(0) = (1, 0, \ldots, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Since $\varphi$ is a diffeomorphism, we can choose a small ball $B$ of the point 0 in the subspace $\mathbb{R}^{m-1}$.

Now by the local flow theorem, there exist a smaller open neighborhood, $V_0 \subset \varphi(V)$ and a function $\psi : (-\varepsilon, \varepsilon) \times V_0 \to \varphi(V)$ such that

$$\psi(0, p) = p,$$
$$\frac{\partial}{\partial t} \psi(t, p) = \varphi \cdot X(\psi(t, p)).$$

In particular, for a small $\varepsilon$, since the ball $B_\varepsilon \subset \mathbb{R}^{m-1}$ could be contained in this $V_0$, the restriction of $\psi$ on $(-\varepsilon, \varepsilon) \times B_\varepsilon$ also satisfies the above conditions. Observe this function $\psi' = \psi|_{(-\varepsilon, \varepsilon) \times B_\varepsilon}$, we have that $D\psi'(0) = I$, the identity map. It follows in a small neighborhood of 0, $\psi'$ is an diffeomorphism. Then we can choose $\varepsilon$ even smaller to make sure $(-\varepsilon, \varepsilon) \times B_\varepsilon$ is contained in this neighborhood. Note that $\varepsilon = \varepsilon_{t_0}$ depends on $t_0$.

The final step is to carefully “link” all these “tubulars” to get one long tubular. Note for each $t_0$, $F_{t_0} = \psi'((-\varepsilon_{t_0}, \varepsilon_{t_0}) \times B_\varepsilon)$ is an open neighborhood of the point $\gamma(t_0)$, it would be a good exercise to prove that when the radius of $B_{\varepsilon_{t_0}}$ is small enough, the intersection $F_{t_0} \cap \gamma([-\varepsilon, \ell + \varepsilon])$ is simply connected, i.e., it is actually $\gamma((-\varepsilon_{t_0}, \varepsilon_{t_0}) \times \{0\})$.

The $F_{t_0}$ form an open cover of the compact set $\gamma([0, \ell])$, so there exists a finite subcover, namely, $F_1, F_2, \cdots, F_k$ where $F_i = \phi_i((-\varepsilon_i, \varepsilon_i) \times B_i)$. Note that the orders of $F_i$’s are very natural, according to the left end points of the corresponding line segments within $(-\varepsilon, \ell + \varepsilon)$.

For $F_1, F_2$ and $\phi_1, \phi_2$, denote that $F_1 \cap \gamma(-\varepsilon, \ell + \varepsilon) = (a_1, b_1)$ and $F_2 \cap \gamma(-\varepsilon, \ell + \varepsilon) = (a_2, b_2)$, where $a_1 < 0 < a_2 < b_1 < b_2$. Now let $\Sigma = \phi_1((-\varepsilon \times B_1)$, then we can make $B_1$ smaller in order that for a small number $\delta$, it holds that $\phi_1((b_1 - \delta) \times B_1)$ is contained in $F_1 \cap F_2$. Then fix any small number $\rho$, we can shrink $B_1$ even more to make that $X_t(\Sigma) \subset F_1 \cup F_2$ for all $t \in (a_1, b_2 - \rho)$. Then let $F_{1,2} = \bigcup_{t \in (a_1, b_2 - \rho)} X_t(\Sigma)$, and $\phi_{1,2} : (a_1, b_2 - \rho) \times B \to F_{1,2}$ is an surjective map, defined as $\phi_{1,2}(t, x) = X_{t-a_1}(\phi_1(a_1, x))$.

Choosing $\rho$ to be small enough, we can suppose $F_{1,2} \cap F_3 \neq \emptyset$. By induction, we can get our final conclusion.

**Corollary 1.3** Let $M$ be a manifold of $C^\infty$ class and $\gamma : [0, 1] \to M$ is an embedding of class $C^\infty$. Then there exists a local chart $\phi : U \to \mathbb{R}^m$ such that the image of $\gamma$ is contained in $U$ and its image under $\phi$ is $[0, 1] \times \{0\} \subset (0, 1) \times \mathbb{R}^{m-1}$.

## 2 Riemannian metrics

A Riemannian metric of class $C^k$ in an open set $U \subset \mathbb{R}^m$ is a map which at each point $x \in U$ to which an inner product is associated as:

$$<,> : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

such that if $X, Y : U \to \mathbb{R}^m$ are vector fields of class $C^k$ then the function $f(x) = <X(x), Y(x)> \in \mathbb{R}$ is of class $C^k$. 

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Let $\frac{\partial}{\partial x_i} : U \to \mathbb{R}^m$ be vector fields $x \to (0, \cdots, 0, 1, 0, \cdots, 0)$ where the $i$-th coordinate is 1. Then the functions $g_{i,j}(x) = \left< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right>_x$ satisfy the conditions
\[
\begin{cases}
g_{i,j}(x) = g_{j,i}(x); \\
(v_1, \cdots, v_m) \cdot \begin{pmatrix} g_{11}(x) & \cdots & g_{1m}(x) \\
\cdots & \cdots & \cdots \\
g_{m1}(x) & \cdots & g_{mm}(x) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\
\cdots \\
v_m \end{pmatrix} > 0 \text{ for any } v \in \mathbb{R}^m \setminus \{0\}.
\end{cases}
\]

On the other hand, if $g_{ij}$ satisfy the above conditions they define a Riemannian metric by the formula
\[
\left< v, w \right>_x = (v_1, \cdots, v_m) \cdot \begin{pmatrix} g_{11}(x) & \cdots & g_{1m}(x) \\
\cdots & \cdots & \cdots \\
g_{m1}(x) & \cdots & g_{mm}(x) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\
\cdots \\
v_m \end{pmatrix}.
\]

If $\alpha : [0, 1] \to U$ is a $C^1$ piecewise $C^1$ curve, define the length of $\alpha$ by
\[
\ell(\alpha) = \int_0^1 \left\| \frac{d}{dt} \alpha(t) \right\|_{\alpha(t)} dt,
\]
where $(\left\| \frac{d}{dt} \alpha(t) \right\|_{\alpha(t)})^2 = < \frac{d}{dt} \alpha(t), \frac{d}{dt} \alpha(t) >_{\alpha(t)}$. A Riemannian metric defines a metric $d : U \times U \to \mathbb{R}$ by
\[
d(x, y) = \inf_{\alpha} \{ \ell(\alpha), \alpha : [0, 1] \to U, \alpha(0) = x, \alpha(1) = y \}
\]
where the infimum is over all the $C^1$ piecewise curves.

The following properties of the metric should be checked:
1. $d(x, x) > 0$.
2. If $x \neq y$, then $d(x, y) > 0$.
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq d(x, y) + d(y, z)$.

We only need to check item (2), while other items are easy. Fix any $x_0 \in U$, we have that the matrix $M = (g_{ij}(x_0))$ is positive definite, so there exists some unitary matrix $P$ such that $P^{-1}DP = M$, where $D$ is diagonal matrix with diagonals $\lambda_1, \cdots, \lambda_m$ (all eigenvalues, actually).

There exists a neighborhood $V_{x_0}$ of the point $x_0$ such that the matrix $(g_{ij}(x))$ for each $x \in V_{x_0}$ these eigenvalues do not change much. So if the neighborhood $V_{x_0}$ is small enough, then there exists some constant $C_{x_0} \geq 1$ such that $\forall x \in \mathbb{R}^m$, we have
\[
\frac{1}{C_{x_0}} \|v\| < \|x\| < C_{x_0}\|v\| \text{ for any } x \in V_{x_0},
\]
where $\|v\|_x = (\left< v, v \right>_x)^{\frac{1}{2}}$.

This fact that these two norms are equivalent implies that the topology of the set $U$ induced from the Riemannian metric is the same with the usual induced topology of $U$ as a subset of $\mathbb{R}^m$. Thus, we have that the function $(x, y) \to d(x, y)$ is continuous and given a compact subset $K \subset U$, there exists a constant $C_K \geq 1$ such that
\[
\frac{1}{C_K} \|x - y\| \leq d(x, y) \leq C_K\|x - y\|.
\]

The item (2) above follows from the above inequalities.

Let $x \in U \to <, >_x : V \to \mathbb{R}$ be two $C^K$ Riemannian metrics and $d, \tilde{d}$ are the corresponding metric functions. A $C^{k+1}$ class diffeomorphism $f : U \to V$ is called an isometry if $\|Df_xv\|_{f(x)} = \|v\|_x$ for any $v \in \mathbb{R}^m$. It is equivalent to the fact that
\[
< Df_xv, Df_xw >_{f(x)} = \langle v, w \rangle_x \text{ for any } v, w \in \mathbb{R}^m.
\]
\[ \tilde{d}(f(x), f(y)) = d(x, y) \text{ for all } v, w \in \mathbb{R}^m \]

Similarly, a $C^k$ class Riemannian metric on $M$ is a map, which, to each $x \in M$, an inner product is associated,

\[ \langle , \rangle : TM_x \times TM_x \to \mathbb{R} \]

such that for any vector fields $X, Y$ in $M$ the function $x \in M \rightarrow \langle X(x), Y(x) \rangle_x$ is of class $C^k$. If \{ $\varphi_i : U_i \subset M \to \tilde{U}_i \subset \mathbb{R}^m$ \} is an $C^k$ atlas in $M$, a Riemannian metric in $M$ can be identified with a family of Riemannian metrics, one for each $\tilde{U}$, such that the changes of coordinates are isometries.

More generally, a symmetric bilinear form of class $C^k$ in $M$ is a function $b$ which for each $x \in M$ assigns a symmetric bilinear form $b(x) : TM_x \times TM_x \to \mathbb{R}$, such that for any vector spaces $X$ and $Y$ of class $C^r$ the function $b(x)(X(x), Y(x))$ is of class $C^k$. Therefore a Riemannian metric is a symmetric bilinear form of class $C^k$ which is positive definite, i.e. $b(x)(v, v) > 0$ for any $v \in TM_x \setminus \{ 0 \}$.

**Proposition.** Each manifold of class $C^{k+1}$ admits a Riemannian metric of class $C^k$.

**Proof.** Let $\varphi_i : W_i \to B(0; 3)$ be a family of local charts such that:

- $W_i$ is a locally finite family,
- $\cup_{i=1}^{\infty} U = M$ where $U_i = \varphi_i^{-1}(B(0; 1))$.

Let $\delta : \mathbb{R}^m \to [0, 1]$ be a $C^\infty$ function which takes value 1 in $B(0; 1)$ and takes 0 outside $B(3)$. Define a symmetric bilinear form $b_i$ in $M$ such that, $b_i(v, w) = \delta(\varphi_i(x)) \langle (D\varphi_i)_x v, (D\varphi_i)_x w \rangle$ for any $x \in W_i$, where $\langle , \rangle$ is a usual inner product in $\mathbb{R}$. So $b_i(x)(v, w) = 0$ if $x \notin W_i$.

If $v \in TM_i \setminus \{ 0 \}$ we can extend $b_i(v, v)$ such that $b_i(v, v) \geq 0$ for any $x \in M$ and $b_i(x)(v, v) > 0$ for any $x \in U_i$. Then $v, \langle v, v \rangle_\tau$ is the desired Riemannian metric in $M$.

**Lemma 2.1** Let $M$ be a $C^k$ manifold which is not compact and $K_i$ be a sequence of compact sets such that $K_i \subset \text{Int}(K_{i+1})$. Let $\epsilon_i > 0$ and $w_i > 0$ be sequences of positive numbers. Then there exist $C^k$ class functions $f, g : M \to \mathbb{R}^+$ such that

\begin{align*}
0 < f(x) &\leq \epsilon_i \text{ for any } x \in K_{i+1} \setminus \text{Int}(K_i), \quad (2.1) \\
g(x) &\geq 0 \text{ for any } x \in K_{i+1} \setminus \text{Int}(K_i). \quad (2.2)
\end{align*}

**Proof.** We only prove the first identity. Firstly, let \{ $\varphi_i$ \} be a partition of unity subordinate to the open cover \{ $\text{Int}(K_j) \setminus K_{j-1}$ \}. Let $\delta_n = \min\{\epsilon_n, \epsilon_{n-1}\}$. Then we claim that the function \( f(x) = \sum_{n=1}^{\infty} \delta_n \varphi_n(x) \) satisfies the condition. In fact, if $x \in K_n \setminus K_{n-1}$, then $\varphi_j(x) \neq 0$ only possibly when $j = n, n+1$. So $f(x) = \delta_n \varphi_n(x) + \delta_{n+1} \varphi_{n+1}(x) \leq \epsilon_n (\varphi_n(x) + \varphi_{n+1}(x)) \leq \epsilon_i$.

**Definition 2.2** A Riemannian metric in $M$ is called complete in $M$, if the corresponding metric function is a complete metric, i.e., all the Cauchy sequences are convergent.

**Proposition.** Any manifold admits a complete Riemannian metric.

**Proof.** Let $\langle , \rangle_x$ be a Riemannian metric in $M$ and $d^1$ the corresponding metric function. Since the metric induces the manifold topology, we have that if $M$ is compact, then the metric is complete.

Now suppose that $M = \cup_{i=1}^{\infty} K_i$ with $K_i \subset \text{Int}(K_{i+1})$, where $K_i$'s are all compact subsets. Since $K_i$ is compact and $d^1$ is continuous there exists some $\tau_i > 0$ such that if $x \in K_i$ and $x \notin K_{i+1}$ then $d^1(x, y) > \tau_i$.

Let $a_i = \max\{1, 1/\tau_i\}$ and $g : M \to \mathbb{R}^+$ be a function of class $C^\infty$ such that $g(x) \geq a_i$ for any $x \in \text{Int}K_i \setminus K_{i-1}$. Consider the Riemannian metric $\langle v, w \rangle_x = g(x) \langle v, w \rangle_1$. If $\alpha : [0, 1] \to M$ be a curve contained in $K_i \setminus \text{Int}(K_{i-1})$ with $\alpha(0) \in \partial K_{i-1}$ and $\alpha(1) \in \partial K_i$. Then we have that
\( \ell^1(\alpha) = \int_0^1 \| \frac{d\alpha}{dt}(t) \|_1^1 dt \geq \tau_i \) (by \( \ell^1(\alpha) \) we mean the length of the curve \( \alpha \) with respect to the metric \( d^1 \))

Then we have \( \ell(\alpha) = \int_0^1 \| \frac{d\alpha}{dt}(t) \|_1^1 dt \geq 1 \). Thus, if \( x \in K_i \) and \( y \in K_{i+p} \) we have that \( d(x, y) \geq p \).

Suppose \( \{x_n\} \) is a Cauchy sequence with respect to the metric \( d \). Then we have \( d(x_0, x_n) \leq N \) for any \( n \geq 1 \) for some \( N \). Thus if \( x_0 \in K_i \), we have that \( x_n \in K_{i+N} \) for all \( n \). So the sequence is contained in a compact subset and the conclusion in the first case lead to the completion of this proof.