1 Partition of unity

Property 1. If $0 < a < b$, then there exists a $C^\infty$ function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\|\varphi(x)\| \leq 1$ and that:

$$\varphi(x) = \begin{cases} 1, & \text{if } \|x\| \leq a, \\ 0, & \text{if } \|x\| \geq b. \end{cases}$$

Proof. First define a $C^\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ as follows,

$$\alpha(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ e^{-\frac{1}{t}}, & \text{if } t \geq 0, \end{cases}$$

which is a function of $C^\infty$ class. Then define $\beta(t) = \alpha(b-t)\alpha(t-a)$, a $C^\infty$ function of t which vanishes when $t \leq a$ or $t \geq b$. Then let $\delta(t) = \int_{a}^{b} \beta(s) ds$, which is a $C^\infty$ function taking values 1 when $t \leq a$ and 0 when $t \geq b$. Finally, define $\lambda(x) = \delta(\|x\|)$.

Property 2. Let $M$ be a $C^\infty$ manifold, then there exists a sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots$ such that $K_j \subset \text{Int}(K_{j+1})$ and $\bigcup_{j=1}^{\infty} K_j = M$.

Note that this property is a consequence of the fact that a topological manifold has a countable basis of open sets.

Proof. We claim that if $\{U_\lambda, \lambda \in \Lambda\}$ is an open cover of $M$ then it has a countable subcover. In fact, let $\{B_n, n \in \mathbb{N}\}$ be a countable basis of open sets of $M$. Then for any $x \in M$ and $x \in U_\lambda$ for some $\lambda$ we have that there exists some $n$ such that $x \in B_n \subset U_\lambda$ (by axiom of choice). Then we have an open cover of the manifold $M$ by the elements of the basis. By construction, each $B_n$ is contained in some $U_\lambda$, so we obtain a countable subcover.

For each $x \in M$ let $V_x$ be a compact neighborhood of $x$ and then the interior of $V_x$ give an open cover of $M$. We can then find a countable subcover $\text{Int}(V_1), \cdots, \text{Int}(V_n), \cdots$ of $M$.

Now let $K_1 = V_1$. As $K_1$ is compact $\text{Int}(V_1)$ covering $K_1$ contain a finite subcover. Let $K_2$ be the union of these $V_j$, which is a compact set whose interior contains $K_1$, we can also make $V_2$ be contained in $K_2$ without loss of other properties. Thus, we can define inductively a sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \cdots$ satisfying the first condition. And since $\bigcup_{j=1} \bigcup_{i=1}^{\infty} U_j = M$, we have $\bigcup_{i=1} K_i = M$, too.

Definition 1.1 Let $f : M \to \mathbb{R}$. The support of $f$, denoted as $\text{supp}(f)$, is the closure of the subset of $M$ where $f$ takes values different from 0.

Definition 1.2 Let $\mathcal{V} = \{V_i; i \in I\}$ be an open cover of $M$. A partition of unity subordinate to the open cover $\mathcal{V}$ is a family $\varphi_i : M \to [0, 1]$ of $C^\infty$ functions such that

1. $\text{supp}(\varphi_i) \subset U_i$.

2. $\{\text{supp}(\varphi_i) \subset U_i; i \in I\}$ is locally finite: any $x \in M$ has a neighborhood $V_x$ such that $\{i; V_x \cap \text{supp}(\varphi_i) \neq \emptyset\}$ is a finite set.

3. $\sum_i \varphi_i(x) = 1$ for any $x \in M$. 

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Note that the sum of item (3) is always over finite terms because of item (2).

**Theorem 1.3** For any given open cover there is a partition of unity subordinate to it.

**Proof.** Let $\mathcal{W} = \{W_\lambda, \lambda \in \Lambda\}$ be an open cover of $M$ which refines the cover $\mathcal{V} = \{V; i \in I\}$, i.e., there exists a function $l : \Lambda \to I$ such that $W_\lambda \subset V_{l(\lambda)}$ for any $\lambda \in \Lambda$. If there exists a partition of unity $\{\phi_\lambda; \lambda \in \Lambda\}$ subordinate to $\mathcal{W}$, then there exists a partition of unity $\{\psi_i; i \in I\}$ subordinate to $\mathcal{V}$. To see this, it suffices to define $\psi(x) = \sum_{l(\lambda) = i} \phi_\lambda(x)$. Note that for any $x \in M$ there are only finite number of $\phi_\lambda$ that do not vanish at $x$, which implies that the number of $\lambda$ with $l(\lambda) = i$ is at most finite, so the summation makes sense.

Let the given any open cover be $\mathcal{A} = \{A_i; i \in I\}$, consider a sequence of compact sets $K_i$ such that $\bigcup_{i=1}^\infty K_i = M$ and $K_i \subset \text{Int}(K_{i+1})$. For each $x_0 \in M$, we choose an open neighborhood $W_{x_0}$ of $x_0$ with the local chart $\varphi_{x_0}$, which satisfies the following conditions:

1. $\varphi_{x_0}(x_0) = 0$, $W_x$ is contained in some element $A \in \mathcal{A}$.
2. $\varphi_{x_0}(W_{x_0}) = B(0;3)$. (we use $B(0;3)$ to denote the open ball in the Euclidean space with the origin as the center and radius 3.)
3. When $x_0 \in K_{i+1} \setminus \text{Int}(K_i)$ (which is a compact set), we choose $W_{x_0}$ carefully to make it contained in $\text{Int}(K_{i+2}) \setminus K_{i-1}$ (which is an open set).
4. Use $U_{x_0}$ to denote the set $\varphi_{x_0}^{-1}(B(0;1))$ and $V_{x_0} = \varphi_{x_0}^{-1}(B(0;2))$, the apply Property 1 to construct a $C^\infty$ function $\lambda_{x_0}$ on $M$ which takes values 1 in $U_{x_0}$ and takes value 0 outside $V_{x_0}$. (just define $\lambda_{x_0}(x) = \lambda \circ \varphi_{x_0}(x)$, where the function $\lambda$ is from property 1 with $a = 1, b = 2$.)

Now we observe the open cover $U_x$’s of the space $M$. Since each $K_{i+1} \setminus \text{Int}(K_i)$ is compact, we can choose finite subcover from these $U_x$’s and therefore, we get a countable open cover $\{U_i, i = 1, 2, \ldots\}$ of $M$ with the corresponding $V_i, W_i$ and $\lambda_i$. By item (1) and the first paragraph above, it suffices to prove our conclusion for the open cover $\{W_i; i \in \mathbb{N}\}$. By construction, $W_i$ is locally finite, this ensures that for any $x \in M$, the number of $\lambda_i$ such that $\lambda_i(x) \neq 0$ is finite. Then we define

$$\phi_i(x) : M \to [0, 1]$$

$$\phi_i(x) = \frac{\lambda_i(x)}{\sum_{j=1}^\infty \lambda_j(x)}$$

$\phi_i$ is a partition of unity subordinate to the open cover $\mathcal{A}$.

**Corollary 1.4** If $K \subset V \subset M$, where $K$ is closed and $V$ is open, then there exists a $C^\infty$ function $\lambda : M \to [0, 1]$ such that $\lambda(x) = 1$ for any $x \in K$ and $\lambda(x) = 0$ if $x \in M \setminus V$.

**Proof.** Since $\{M \setminus K, V\}$ is an open cover of $M$, then the conclusion of this Corollary is an easy consequence of the theorem.

## 2 Vector fields in manifolds

A vector field $C^r$ in an open set $U \subset \mathbb{R}^m$ is a $C^r$ map $X : U \to \mathbb{R}^n$. Let $\alpha : I = [a,b] \subset \mathbb{R} \to U$ be an integral curve of $X$ if $\frac{d}{dt} \alpha(t) = X(\alpha(t))$. Then $\alpha(I)$ is called an orbit of the vector space $X$.

**Theorem 2.1** 1. For each $x \in U$ there exists an integral curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = x$.
2. If $\alpha : I \to U$ and $\beta : J \to V$ are integral curves of $X$ and $\alpha(t_0) = \beta(t_0)$ then $\alpha(t) = \beta(t)$ for any $t \in I \cap J$. 

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3. For each $x_0 \in U$ there exists a neighborhood $V \subset U$ of $x_0$ and $\varepsilon > 0$ and a $C^r$ function $\varphi : (-\varepsilon, \varepsilon) \times V \to U$, such that
(a). $\varphi(0, x) = x$ for any $x \in V$.
(b). $\frac{\partial \varphi}{\partial t}(t, x) = X(\varphi(t, x))$.

The extension of this concept to $C^\infty$ manifolds Let $f : U \to W$ be a diffeomorphism of class $C^{k+1}$ between open sets of $\mathbb{R}^m$, $X : U \to \mathbb{R}^m$ and $Y : W \to \mathbb{R}^m$ are $C^k$ vector fields then the following two conditions are equivalent.

1. For any integral curve $\alpha$ of $X$, $f \circ \alpha$ is an integral curve of $Y$.
2. $X = f^*Y$, where $(f^*Y)(x) = (Df(x))^{-1}Y(f(x))$.

Under these conditions, we say that the vector space $X = f^*Y$ is the pull-back of $Y$ under diffeomorphism $f$.

Definition 2.2 Let $\{\varphi : U_i \subset M \to \widetilde{U}_i \subset \mathbb{R}^m \mid i \in I\}$ is a $C^r$ atlas with $r \geq k + 1$, in a manifold $M$. A $C^k$ vector field $X$ on $M$ is a family of $C^k$ vector fields $X_i : \widetilde{U} \to \mathbb{R}^m$ such that

$$(\varphi_j \circ \varphi_i^{-1})^*(X_j|\varphi_j(U_i \cap U_j)) = X_i|\varphi_i(U_i \cap U_j)$$

By this condition, for any $x \in M$ there exists a unique $X(x) \in TM_x$ such that $d\varphi_i(x)X(x) = X_i(\varphi_i(x))$.

Proposition. If $\alpha, \beta : I \to M$ are integral curves of some $C^r$ vector space $X$ and $\alpha(t_0) = \beta(t_0)$. Then $\alpha(t) = \beta(t)$, $\forall t \in I$.

Proof. By the local uniqueness of solutions, the set $\{t|\alpha(t) = \beta(t)\}$ is an open set. Note this set is also closed by property of these two functions. Since $t_0 \in \{t|\alpha(t) = \beta(t)\}$, it is not an empty set, it must be the whole set $I$.

Definition 2.3 Let $\alpha : (w_-, w_+) \subset I \subset \mathbb{R} \to M$ be an integral curve of $X$. $(w_-, w_+)$ is the maximal interval if for any integral curve $\beta : I \to M$ with $\beta(t_0) = \alpha(t_0)$ we have that $I \subset (w_-, w_+)$.

Proposition. Let $X$ be a vector field of class $C^r$, $r \geq 1$ on a manifold $M$. Let $\alpha : (w_-, w_+) \to M$ be an integral curve in its maximal interval. If $w_+ < \infty$ then for each compact $K \subset M$ there exists some $\varepsilon > 0$ such that when $t \geq w_+ - \varepsilon$, $\alpha(t) \notin K$. There are, of course, similar statement for $w_-$.

Proof. Suppose for contradiction that there exists a sequence $t_n \to w_+$ such that $X(t_n) \in K$. Since $K$ is compact, passing to a subsequence if necessary, we can suppose that $\alpha(t_n) \to x \in K$. Thus, there exists some $\varepsilon > 0$, a neighborhood $V$ of $x$ and a $C^k$ differentiable function $\varphi : (-\varepsilon, +\varepsilon) \times V \to M$ such that for any $y \in V$ the map $t \in (-\varepsilon, +\varepsilon) \to \varphi(t, y)$ is an integral curve of $X$ with $\varphi(0, y) = y$.

Now we can take some $y = \alpha(t_n)$ such that $w_+ - \varepsilon < t_n$.

Now define $\widetilde{\alpha} : (w_-, t_n + \varepsilon) \to M$ defined by $\widetilde{\alpha}(t) = \alpha(t)$ when $t < w_+$ and $\widetilde{\alpha}(t) = \varphi(t - t_n, y)$ when $t \in (t_n - \varepsilon, t_n + \varepsilon)$. Thus $\widetilde{\alpha}$ defines an integral curve on $X$ with $\widetilde{\alpha}(t_n) = \alpha(t_n)$. Then the interval $(w_-, t_n + \varepsilon)$ contains $(w_-, w_+)$ strictly, which is the desired contradiction.

Definition 2.4 $X$ is called a complete vector field if each integral curve of $X$ is defined on $\mathbb{R}$. If $X$ is a complete vector field, there exists a $C^r$ function $\varphi : \mathbb{R} \times M \to M$ such that $\varphi(0, x) = x$ and that $\frac{\partial \varphi}{\partial t}(t, x) = X(\varphi(t, x))$. By the theorem of solution to the differentiable equations with respect to initial conditions, we have that the function $\varphi$ is of class $C^k$. The map $\varphi$ is thus called the flow of the field $X$. 3
Let \( s \in \mathbb{R} \) and \( \Phi : \mathbb{R} \times M \to M \) be the flow of the field \( X \). As the curves \( \varphi : \mathbb{R} \to M, \beta : \mathbb{R} \to M \) defined by
\[
\alpha(t) = \Phi((t + s), x) \\
\beta(t) = \Phi(t, \Phi(s, x))
\]
are integral curves of \( X \) and \( \alpha(0) = \beta(0) \). So \( \Phi((t + s), x) = \Phi(t, \Phi(s, x)) \) for any \( t \). Define \( X_t : M \to M \) be the \( C^k \) map defined by \( X_t(x) = \Phi(t, x) \). We have that \( X_0 = Id \) and \( X_{s+t} = X_s \circ X_t \). In particular, \( X_{-t} \circ X_t = Id \) and therefore \( X_t \) is a diffeomorphism of \( M \).