1 Differentiable manifold of dimension n

Definition 1.1 (topological manifold) Given a topological space satisfying the following conditions: 1. Hausdorff; 2. With countable basis of open sets (separable); 3. given $x \in M$, there exists some open set $U \subset M$ containing $x$, $\bar{U} \subset \mathbb{R}^n$ and a homeomorphism $\psi : U \rightarrow \bar{U}$ which is called a local chart of $M$. Note that there exists a family $\psi_\alpha : U_\alpha \subset M \rightarrow \bar{U}_\alpha \subset \mathbb{R}^n$ (each $\psi_\alpha$ is homeomorphism), and $U_\alpha$ cover the whole $M$. Hence, any $\psi_\beta \circ \psi_\alpha^{-1}$, restricted on $U_\alpha \cap U_\beta$, is a homeomorphism, and $A = \{U_\alpha, \psi_\alpha\}$ is called an atlas on $M$.

Definition $A = \{U_\alpha, \psi_\alpha\}$ is a $C^r$ atlas if all the transition functions are $C^r$ diffeomorphisms; $A$ is a holomorphic atlas if all the transition functions are holomorphic diffeomorphisms.

We say two atlas $A_1$ and $A_2$ are equivalent if $A_1 \cap A_2$ is also an $C^r$ atlas. By zorn’s lemma, there is a unique maximal $C^r$ atlas containing a given atlas $A$, and it is called the $C^r$ differential structure.

Examples
1. All the open sets in $\mathbb{R}^n$ is a $C^\infty$ structure.
2. $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a $C^r$ map, $y \in \mathbb{R}^p$. $y$ is a regular value of $f$, i.e., for any $x \in U$ such that $f(x) = y$ we have that $df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is surjective. Then either $f^{-1}(y) = \emptyset$ or $f^{-1}(y)$ has a $C^\infty$ differential structure of dimension $n - p$. (This statement is actually a result of the local submersion theorem)
3. $S^n = \{x = x_1, x_2, \ldots, x_{n+1} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$. Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ defined as $f(x) = \sum_{i=1}^{n+1} x_i^2$. Then if $x \neq 0$, $df(x) \cdot h = \sum_{i=0}^{n+1} 2x_i h_i$. So the fact that 1 is a regular value implies that $f^{-1}(1)$ is manifold of dimension $n$.
4. (Real Projective Spaces) $\mathbb{R}P^n$, as a set, is all the lines containing the origin in $\mathbb{R}^{n+1}$. This means that $\mathbb{R}P^n = \mathbb{R}^{n+1}/\sim$, where the equivalence relation $\sim$ is defined as:

$$x \sim y \iff \text{there exists some } \lambda \in \mathbb{R}\setminus\{0\} \text{ such that } y = \lambda x$$

(Note that we can also regard it as the quotient space $\mathbb{R}P^n = S^n/\sim$, where $\sim$ is the equivalence relation on $S^n : x \sim -x$)

In both cases, the topology of $\mathbb{R}P^n$ is the induced topology, i.e., let $q : X \rightarrow X/\sim$ be the transformation mapping each point of $X$ to its equivalence class $[x] \in X/\sim$. then:

$$U \subset X/\sim \text{ is an open set } \iff q^{-1}(U) \text{ is an open set}$$

Let’s study the structure of $\mathbb{R}P^n$. Firstly, denote the equivalence class the point $(x_1, \ldots, x_{n+1})$ by $[x_1, \ldots, x_{n+1}]$. Then we have $[x_1, \ldots, x_{n+1}] = [\lambda x_1, \ldots, \lambda x_{n+1}]$ for any $\lambda \neq 0$. Note that $U_i = \{x_1, \ldots, x_{n+1} \}, x_i \neq 0$ is open, since the preimage of it under the map $q$, $V_i = q^{-1}U_i = \{x \in \mathbb{R}^{n+1}/\{0\} \}$ is an open subset.

Define $\varphi_i : U_i \rightarrow \mathbb{R}^n$ as $\varphi_i([x_1, \ldots, x_{n+1}]) = \varphi_i((\frac{x_1}{x_i}, \ldots, \frac{x_i-1}{x_i}, 1, \frac{x_{i+1}}{x_i}, \frac{x_n}{x_i})) = (\frac{x_1}{x_i}, \ldots, \frac{x_i-1}{x_i}, \frac{x_{i+1}}{x_i}, \frac{x_n}{x_i})$.
Then, in $U_i \cap U_j$, $\varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ should be (supposing that $j > i$):

$$(x_1, \ldots, x_n) \mapsto \varphi_j([x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n])$$

$$= \varphi_j([\frac{x_1}{x_j}, \ldots, \frac{1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{x_{i+1}}{x_j}, \ldots, \frac{x_n}{x_j}])$$

$$= (\frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{x_{i+1}}{x_j}, \ldots, \frac{x_n}{x_j})$$

Observing that $\varphi_j(U_i \cap U_j) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_j \neq 0\}$ and $\varphi_j(U_i \cap U_j) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_j \neq 0\}$, with the fact $f(x) = \frac{1}{x}$ is analytical at each $x \neq 0$, we actually proved that $RP^n$ is a real analytical differentiable manifold.

5. The complex projective space, $\mathbb{C}P^n$, is the set of all the complex lines through the origin in $\mathbb{C}^{n+1}$

$\mathbb{C}P^n = \mathbb{C}^{n+1}/\sim$, where the equivalence relation $\sim$ is defined as:

$$x \sim y \iff \text{there exists some } \lambda \in \mathbb{C}/\{0\} \text{ such that } y = \lambda x$$

The topology of $\mathbb{C}P^n$ is the induced topology, just as in the real projective space, explained in the previous notes. Let $q : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/\sim$ be the projection map, such that $p(z) = [z]$.

Similarly with real projective spaces, we have local charts $U_i = \{[z_1, \ldots, z_{n+1}] : z_i \neq 0\}$, and $\varphi_i : U_i \to \mathbb{R}^n$ as $\varphi_i([z_1, \ldots, z_{n+1}]) = \varphi_i([\frac{z_1}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{z_{i+1}}{z_j}, \ldots, \frac{z_{n+1}}{z_j}])$. Then we have that, in $U_i \cap U_j$, the map $\varphi_j \circ \varphi_i^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ (letting $j > i$):

$$(z_1, \ldots, z_n) \mapsto (\frac{z_1}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{z_{i+1}}{z_j}, \ldots, \frac{z_n}{z_j})$$

is holomorphic and thus $C^\infty$. So $\mathbb{C}P^n$ is a differentiable manifold of complex dimension $n$.

6. Quaternion projective spaces, $\mathbb{H}P^n$, is the quotient space $([\mathbb{H}^{n+1}/\{0\}])/\sim$.

**Definition 1.2 (differentiable transformations)** $M$ is a $C^r$ differentiable manifold of dimension $n$; while $P$ is a $C^r$ differentiable manifold of dimension $p$. Let $f : M \to P$. Let $f : M \to P$ be differentiable at $x_0 \in M$ if there exists local charts such that

$$\varphi : U \subset M \to \tilde{U} \subset \mathbb{R}^n,$$

$$\psi : V \subset M \to \tilde{V} \subset \mathbb{R}^p,$$

such that

$$f(U) \subset V \text{ and } \psi \circ f \circ \varphi^{-1} : \tilde{U} \to \tilde{V} \text{ is differentiable at } \varphi(x_0)$$

By carefully drawing commutative diagrams, one has that the above definition does not depend on the choice of different charts containing the point $x_0$. We define that a transformation $f : M \to P$ is $C^r$ differentiable if it is $C^r$ at each point $x \in M$. And, $f$ is diffeomorphism if there exists some $C^r$ map $f^{-1} : P \to M$ such that $f \circ f^{-1} = Id_P$ and $f^{-1} \circ f = Id_M$.

**Property**. The composition of two $C^r$ functions $f : M \to P$ and $g : P \to N$, $g \circ f : M \to N$, is also $C^r$ function.

Given $x, v \in \mathbb{R}^n$, define $\phi(t) = x + tv$ then we have $\phi(0) = x, \phi'(0) = v$. $v$ can be regarded as the equivalence class of the curves in $\mathbb{R}^n$ through the point $x$ and is tangent with the vector $v$. Let $\alpha, \beta : (-\epsilon, \epsilon) \to M$ be differentiable curves with $\alpha(0) = \beta(0) = x$. Then $\alpha \sim \beta \iff$ there exists a local chart $\phi : U \subset M \to \tilde{U} \subset \mathbb{R}^n$ such that $\phi(x) = 0, (\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0)$. Note that this property does not depend on the choice of the chart, then
Definition 1.3 An equivalence class of differentiable curves through a point $x$ is a tangent vector of $M$ at $x$. The tangent space of $M$ at a point $x \in M$, denoted as $TM_x$, is a vector space, consisting of all the tangent vectors of $M$ at point $x \in M$.

Note that there exists a bijection between $TM_x$ and $\mathbb{R}^n$. Consider a local chart $\psi : U \subset M \to \tilde{U} \subset \mathbb{R}^n$ with $\psi(x) = 0$. Then define $D\psi(x) : TM_x \to \mathbb{R}^n$ by sending the equivalence class $[\alpha]$ to $(\psi \circ \alpha)'(0)$. This bijection shows that $TM_x$ is a vector space of dimension $n$. This definition can be verified to be well defined and indeed a bijection.

To see it is well defined, we observe that if $\phi : U' \to \tilde{U}'$ is another local chart containing $x$, then $D\phi(x) = D(\phi \circ \psi^{-1})(\psi(x)) \circ D\psi(x)$, since $D(\phi \circ \psi^{-1})(\psi(x))$ is an isomorphism, we have that if $D\psi(x)$ being an isomorphism is equivalent with $D\phi(x)$ being an isomorphism. To see it is a bijection, we only need to show it is surjective. For any $v \in \mathbb{R}^n$, define $\alpha(t) = \psi^{-1}(\psi(x) + tv)$, (or, just $\alpha(t) = \psi^{-1}(tv)$, since we supposed $\psi(x) = 0$ in the previous paragraph.) $D\psi(x) = (\psi \circ \alpha)'(0) = v$.

Definition 1.4 Given a differentiable map $f : M^n \to P^p$, $Df : TM_x \to TP_{f(x)}$ is defined as $Df([\alpha]) = [f \circ \alpha]$.

In order to verify that this definition is well defined, it suffices to note that given the choice of representative of the equivalence class does not affect the value of the function. Then suppose $\varphi : U \subset M \to \tilde{U} \subset \mathbb{R}^n$ and $\psi : V \subset P \to \tilde{V} \subset \mathbb{R}^p$ such that $x \in U$ and $f(x) \in V$. We have the following

$$Df(x) = [D\psi(f(x))]^{-1} \circ D(\psi \circ f \circ \varphi^{-1})(\varphi(x)) \circ D\psi(x)$$

So $Df$ is a linear transformation. The chain rule can also be extended to the transformations between manifolds, i.e., suppose $f : M \to P$ and $g : P \to N$, where $f$ is a diffeomorphism at $x$ and $g$ is a diffeomorphism at $y = f(x)$, then $g \circ f$ is a diffeomorphism at $x$ and $D(g \circ f)(x) = Dg(f(x))Df(x)$. The proof is similar with the above proof of the linearity of $Df$ and is omitted.

Definition 1.5 Let $f : M \to P$ be a $C^r$ diffeomorphism, $r \geq 1$. We say $f$ is an immersion if for any $x \in M$, $Df(x) : TM_x \to TP_{f(x)}$ is injective at each point. If $Df(x)$ is surjective, then $f$ is called a submersion. Finally, if an immersion $f$ is a homeomorphism onto its image, then $f$ is an embedding.

An typical example for a 1-1 immersion yet not embedding is the map from the real line to the figure "9", which is not homeomorphism, although 1-1, since the induced topology from the real line is not the same with the subspace topology in its image.

Definition 1.6 A $C^r$, $r \geq 1$ differentiable map, $f : M \to P$, $y \in P$ is called a regular value of $f$, if for any $x = f^{-1}(y)$ the derivative $Df(x)$ is surjective.

Corollary 1.7 Given the above function, if $y$ is a regular value of $f$ and $f^{-1}(y) \neq \emptyset$, then $f^{-1}(y)$ has the $C^r$ differentiable structure of dimension $n - p$. Moreover, the inclusion map $i : f^{-1}(y) \to M$ is a $C^r$ embedding.

From now on we will only consider $C^\infty$ manifolds, because of the following theorem (which will be proved later):

Theorem 1.8 If $M$ is a $C^r$, $r \geq 1$ differentiable manifold, then there exists a $C^\infty$ manifold $N$ and a $C^r$ diffeomorphism $f : M \to N$. 

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