ON FLUSHED PARTITIONS AND CONCAVE COMPOSITIONS

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Abstract. In this work, we give combinatorial proofs for generating functions of two problems, i.e. flushed partitions and concave compositions of even length. We also give combinatorial interpretation of one problem posed by Sylvester involving flushed partitions and then prove it. For these purposes, we first describe an involution and use it to prove core identities. Using this involution with modifications, we prove several problems of different nature, including Andrews’ partition identities involving initial repetitions and partition theory interpretations of three mock theta functions of third order \( f(q) \), \( \phi(q) \) and \( \psi(q) \). An identity of Ramanujan is proved combinatorially. Several new identities are also established.

Keywords: Integer partition, flushed partition, concave composition, involution, mock theta function.

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1 Introduction

In this paper, we are mainly concerned with two problems in the theory of integer partition, namely, flushed partitions and concave compositions of even length.

The definition of flushed partition is given by Sylvester [16]. A partition is called flushed when the number of the parts with length \( k \) is odd, where \( k = 1, 2, \ldots, 2i - 1 \), and the number of parts with length \( 2i \) do not occur odd number of times. Similarly, define unflushed partitions as those not satisfying the above conditions. Sylvester also posed two problems with respect to flushed partitions. One of them, in Sylvester’s words, is stated as follows,

“1. Required to prove, that if any number be partitioned in every possible way, the number of unflushed partitions containing an odd number of parts is equal to the number of unflushed partitions containing an even number of parts.

“Ex.gr.: The total partitions of 7 are 7; 6, 1; 5, 2; 5, 1, 1; 4, 3; 4, 2, 1; 4, 1, 1, 1; 3, 3, 1; 3, 2, 2; 3, 2, 1, 1; 2, 2, 2, 1; 3, 1, 1, 1, 1; 2, 2, 1, 1, 1; 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1. Of these, 6, 1; 4, 1, 1, 1, 1; 3, 3, 1; 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1, 1 alone are flushed. Of the remaining unflushed partitions, five contain an odd number of parts, and five an even number.

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“Again, the total partitions of 6 are 6; 5, 1; 4, 2; 4, 1, 1; 3, 3; 3, 2, 1; 2, 2, 2; 3, 1, 1, 1; 2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1; of which 5, 1; 3, 2, 1; 3, 1, 1, 1 alone are flushed. Of the remainder, four contain an odd and four an even number of parts.

“N.B.—This transcendental theorem compares singularly with the well-known algebraical one, that the total number of the permuted partitions of a number with an odd number of parts is equal to the same of the same with an even number.

Solution to this problem is given by Andrews in 1970 in Andrews [2] by manipulating generating functions, and the generating function for flushed partitions reads as follows,

$$\frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} q^{n(3n-1)/2} (1 - q^n).$$  \hspace{1cm} (1.1)

However, Andrews doubted that his proofs are what Sylvester expected in the first place. He writes (in Andrews [8]): “It is completely unknown whether this was Sylvester’s approach and how he came upon flushed partitions in the first place.”

It is hardly believed that the core of the combinatorial proof of Sylvester’s problem is only one involution. But it turns out to be the case. We will first prove combinatorially the above generating function (1.1), and then by inserting another variable $z$ into the generating function, we can map bijectively unflushed partition of $n$ with $m$ parts into unrestricted partitions of $n$ with $m$ parts with additional restrictions involving Durfee Symbols, a very natural concept yet just introduced recently by Andrews [4]. In this way, we try to understand flushed partitions in a new combinatorial sense and then provide a new proof of Sylvester’s problem.

**Concave composition of even length** was recently introduced by Andrews in the study of orthogonal polynomials, see Andrews [7, 9]. It is a sum of the form $\sum a_i + \sum b_i$ such that

$$a_1 > a_2 > \cdots > a_m = b_m < b_{m-1} < \cdots < b_1,$$

where $a_m \geq 0$, and all $a_i$ and $b_i$ are integers. Let $\mathcal{CE}(n)$ denote the set of concave compositions of even length of $n$, and let $ce(n)$ be the cardinality of $\mathcal{CE}(n)$. By transformation formulas, Andrews derived the generating function of $ce(n)$ as follows [Andrews [7], Theorem 1].

For $|q| < 1$,

$$\sum_{n=0}^{\infty} ce(n)q^n = \frac{1}{(q)_{\infty}} \left(1 - \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n)\right).$$  \hspace{1cm} (1.2)

Andrews [7] asked for a combinatorial proof of Theorem 1.2. We will give one such proof in this paper.

Note the above two generating functions (1.1) and (1.2) have connections with one another, which lead to our main theorem as stated as follows.
Theorem 1.1 The number of unflushed partitions of $n$ is equal to the number of concave compositions of even length of $n$.

This paper will be organized as follows. In section 2, we state and prove an involution, which, with various modifications, will be used repeatedly. In Section 3, we give combinatorial proofs of several different q-series identities. Readers who are not interested in these problems can skip directly to section 4, where we will prove our main theorem about these two generating functions and prove Sylvester’s first problem.

The applications of the involutions in section 3 consists of several different partition identities. The nature of these problems varies, showing such involutions are indeed useful tools.

One application is on Andrews’ partition identity involving initial repetitions. In [6], Andrews proved the q-series identity

$$
\sum_{n=0}^{\infty} \frac{z^n q^{1+2+2^2+\ldots+n^2}}{(q)_n} \prod_{j=n+1}^{\infty} (1 - zq^j) = \sum_{j=0}^{\infty} (-1)^j z^j q^{j(j+1)\frac{1}{2}} \tag{1.3}
$$

which, when interpreted combinatorially, means that partitions of $n$ with $m$ different parts and an even number of distinct parts in which, if part $j$ is repeated, then all parts smaller than $j$ is repeated, are equinumerous with partitions of $n$ with $m$ different parts and an odd number of distinct parts satisfying the same property, unless $n$ is a triangular number $\frac{j(j+1)}{2}$ and $m = j$, when their difference is $(-1)^j$. Partitions with this property are called partitions with initial 2-repetitions.

This result (1.3) looks similar to classical Euler’s pentagonal number theorem. The involution plays a role as the role the well known Franklin’s involution has played in Euler’s pentagonal number theorem. Based on this proof, a formula is given to compute the number of partitions into even number of distinct parts.

The next applications involve combinatorial interpretations of the following three mock theta functions of order 3, defined by Ramanujan,

$$
f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)^n} \tag{1.4}
$$

$$
\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n} \tag{1.5}
$$

$$
\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n} \tag{1.6}
$$
The combinatorial interpretation of (1.4) and (1.5) are given in the end of section 2. The following combinatorial interpretation of (1.6) was first given by Fine [15]:

\[ f(q) = 1 + \frac{1}{(-q)\infty} \sum_{k \geq 1} (-1)^{k-1} q^k (-q^{k+1})\infty \]  

(1.7)

We will restate this result as theorem 3.7. Our proof is new.

Then all these combinatorial interpretations lead to the following identity of Ramanujan:

\[ f(q) = \phi(-q) - 2\psi(-q) \]

We will restate this as Corollary 3.8. The first proof of this identity is given by Watson [17]. Another combinatorial proof was given by W. Y. C. Chen, K. Q. Ji, and E. H. Liu [12].

At the end of section 3, we use the involution to generate several more identities, which all have partition interpretations. We only write such interpretation of one identity as an example.

2 An Involution \( \alpha \)

All notations in the theory of integer partitions follow the book Andrews[4]. A partition of a positive integer \( n \) is a finite nonincreasing sequence of positive integers \( (\lambda_1, \cdots, \lambda_r) \) such that \( \sum_{i=1}^{r} \lambda_i = n \). \( \lambda_i \) are called the parts of the partition. We use \( P \) to denote the set of partitions and \( P_n \) to denote the set of partitions of \( n \). We can represent a partition as its Ferrers diagram, that is, a pattern of left-justified boxes with \( \lambda_i \) squares in row \( i \). The square in the \( i \)th row and the \( j \)th column can be written simply as the square \((i, j)\). The Durfee square in \( \lambda \) is the largest square of boxes contained in the partition \( \lambda \). The conjugate of \( \lambda = (\lambda_1, \cdots, \lambda_r) \) is a partition \( \lambda' \) with the \( i \)th part \( \lambda'_i \) as the number of parts of \( \lambda \) that are \( \geq i \).

Define the set of partitions into distinct parts which may contain one copy of empty part as \( D' \). Naturally, \( D'_n \) would denote the set of such partitions of \( n \). Similarly, we define \( P' \) as the set of partitions which can contain empty parts (maybe more than one copies). \( P'_n \), of course, will denote the set of such partitions of \( n \).

We also adopt the following standard abbreviations for q-series:
\((a; q^k)_n = (1 - a)(1 - aq^k) \cdots (1 - aq^{(n-1)k})\),
\((a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\),
\((a; q^k)_\infty = \prod_{n=0}^{\infty} (1 - aq^{nk})\),
\((a)_0 = 0\).

Now we are ready to state the promised involution. We denote it as \(\alpha\). Given a triple \((\lambda, \mu, \rho_{n+\ell})\) satisfying the following properties with a sign \((-1)^{n+k}\).

1. \(\lambda\) has \(n\) parts. The least part has length \(k\). Note that it may contain empty part.
2. \(\mu\) is a partition with no more than \(\ell\) parts.
3. \(\rho_{n+\ell}\) is Sylvester’s triangle \((n + \ell, n + \ell - 1, \cdots, 1)\).

Let the involution \(\alpha\) act on such a triple by comparing \(\lambda_1\) the largest part of \(\lambda\) and \(\mu_1\) the largest part of \(\mu\), as follows.

If \(\lambda_1 \geq \mu_1\), then we move the first part of \(\lambda\) and attach it to \(\mu\), making it \(\mu'\). What has been left there is then \(\lambda'\). Since \(\mu'\) has no more than \(\ell + 1\) parts, yet the smallest part of \(\lambda\) doesn’t change, the sign changes.

On the other hand, if \(\lambda_1 < \mu_1\), then we move the first part of \(\mu\) and attach it to \(\lambda\), making it \(\lambda'\). What has been left there is then \(\mu'\) and \(\mu'\) has no more than \(\ell - 1\) parts, yet the smallest part of \(\lambda\) doesn’t change, the sign changes also in this situation.

The fixed points of this involution are the triples \((\lambda, \mu, \rho_{n+\ell})\) where \(\lambda\) has only one part of length \(t\) and \(t \geq \mu_1\); \(\mu\) is a partition with no more than \(d - 1\) parts.

**Remark 1.** For the triple \((\lambda, \mu, \rho_{n+\ell})\), sometimes the number of partitions of \(\mu\) is strictly less than \(\ell\). This situation makes no exceptions of our arguments. In fact, \((\lambda', \mu', \rho_{(n+\ell)+(\ell+1)}) = \alpha((\lambda, \mu, \rho_{n+\ell}))\) where \(\ell\) changes by \(\pm 1\).

**Remark 2.** The nature of this involution has a more general form in B.C. Bessenrodt and I. Pak [11].

For example, Let \(\lambda = (9, 6, 5, 2), \mu = (6, 4, 4), k = 2, n = 4, \ell = 3\). It is assigned with \((-1)^6 = -1\). Then we have that \(\alpha((\lambda, \mu, \rho_{4+3})) = (\lambda', \mu', \rho'_{3+4})\), where \(\lambda' = (6, 5, 2), \mu' = (9, 6, 4, 4), k = 2, n' = 3, \ell' = 4\). It is assigned with \((-1)^6 = 1\). We illustrate this example in Figure 1.

Applying this involution, we can prove the following identity:

**Theorem 2.1**

\[
\sum_{n=1}^{\infty} \frac{q^{n^2}(q^{n+1})}{(q)_{n-1}(1 + q^n)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}}{(1 + q) \cdots (1 + q^n)}. \tag{2.1}
\]
the partition $\lambda = (9, 6, 5, 2)$  

the partition $\mu = (6, 4, 4)$ with $\rho_7$  

the partition $\lambda' = (6, 5, 2)$  

the partition $\mu' = (9, 6, 4, 4)$ with $\rho_7$

**Figure 1:** The involution $\alpha$

**Proof.**

For each term in the left hand side of (2.1), we interpret $q^n$, $\frac{1}{(q)_{n-1}}$, $\frac{1}{1+q^n}$ and $(q^n+1)_\infty$ as follows respectively:
1. A $n \times n$ squares.
2. A partition with the largest part of length at most $n - 1$.
3. $k$ parts of length exactly $n$, assigned with $(-1)^k$.
4. A partition with distinct parts and the least part is larger than $n$. Suppose this partition has $\ell$ parts. It is assigned with $(-1)^\ell$.

We split the $n \times n$ squares to a Sylvester’s triangle $\rho_n = (n, n-1, \cdots, 1)$ and another Sylvester’s triangle $\rho_{n-1} = (n-1, n-2, \cdots, 1)$. Then we glue together three objects to form a new partition $\lambda^*$, which are, the $k$ parts of length $n$, $\rho_{n-1}$ and the partition with the largest part of length at most $n - 1$.

Now we observe its conjugate and denote it as $\lambda$. It is a partition with distinct parts, $k$ the length of the least part. Obviously, $k = 0$ is allowed, so the new partition formed can have empty part. That is, $\lambda \in D'$.

Then we attach the Sylvester’s triangle $\rho_n$ under the partition with $\ell$ parts, the least length of parts larger than $n$. This is clearly a partition with distinct parts. We divide it into a partition $\mu$ and Sylvester’s triangle $\rho_{n+\ell}$ in the obvious way. We see that $\mu$ has no more than $\ell$ parts.

Remember this was assigned with $(-1)^{k+\ell}$.

We then invoke the involution $\alpha$ to conclude. Since the fixed points of this involution are the triples $(\lambda, \mu, \rho_d)$ where $\lambda$ has only one part of length $t$ and $t \geq \mu_1$; $\mu$ is a partition
with no more than \( d - 1 \) parts.

We move this \( t \) squares and attach it into \( \mu \) together, getting a partition with no more than \( d \) parts, which is assigned with \((-1)^{t+d-1}\), \( t \) the largest part of \( \mu \), \( d \) is the subscript of the Sylvester Triangle. This correspond to the right-hand side of the identity, which completes the proof.

See figure 2 for a concrete example. The graph in the left is the partition \((13, 10, 9, 4, 4, 4, 4, 4, 3, 3, 1, 1)\), while the graph in the right is \( \lambda = (9, 6, 5, 2), \mu = (6, 4, 4) \) and \( \rho_{4+3} \).

![Figure 2: representation of left hand side of (2.1)](image)

By inserting another variable \( z \) into the identity to track the number of parts or to track the largest part (when observed conjugately, they are the same) and then invoke the above involution, we can generalize Theorem 2.1 into the following two identities:

**Corollary 2.2**

\[
\sum_{n=1}^{\infty} \frac{z^n q^n (z q^{n+1})_\infty}{(q)_{n-1}(1 + q^n)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1 + q)(1 + q^2) \cdots (1 + q^n)}. \tag{2.2}
\]

\[
\sum_{n=1}^{\infty} \frac{z^n q^n (z q^{n+1})_\infty}{(q)_{n-1}(1 + z q^n)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n q^{\frac{n(n+1)}{2}}}{(1 + z q)(1 + z q^2) \cdots (1 + z q^n)}. \tag{2.3}
\]

**Proof.**

In (2.2) and in (2.3), let \( z \to -q^{-1}, q \to q^2 \), we have the following:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n q^{2n^2-n}(-q^{2n+1}; q^2)_\infty}{(q^2; q^2)_{n-1}(1 + q^{2n})} = -\sum_{n=1}^{\infty} \frac{q^{2n^2}}{(-q^2; q^2)_n} = 1 - \phi(q). \tag{2.4}
\]
\[
\sum_{n=1}^{\infty} \frac{(-1)^n q^{2n^2-n}(-q^{2n+1}; q^2)_\infty}{(q^2; q^2)_{n-1}(1-q^{2n})} = -\sum_{n=1}^{\infty} \frac{q^n}{(q; q^2)_n} = -\psi(q). \tag{2.5}
\]

where \(\phi(q) = \sum_{n=1}^{\infty} \frac{q^n}{(-q^2 q^n)_n} \) and \(\psi(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q q^2)_n} \) are two mock theta functions of third order defined by Ramanujan in his last letter to Hardy.

Then by simple calculations with Euler’s identity \((-q)_{\infty} = \frac{1}{(q q^2)_{\infty}}\), we have

\[
(-q; q)_{\infty} (1 - \phi(-q)) = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-1}(1+q^{2n})}. \tag{2.6}
\]

\[
(-q; q)_{\infty} (-\psi(-q)) = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-1}(1+q^{2n})} = \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q)_{2n-2}(1-q^{2(2n-1)})}. \tag{2.7}
\]

We now use \(D_{e,e}(n)\) to denote the number of partitions of \(n\) into even number of distinct parts in which the smallest part is even. \(D_{e,o}(n)\) denote the number of partitions of \(n\) into even number of distinct parts in which the smallest part is odd.

Symmetrically, \(D_{o,e}(n)\) (respectively, \(D_{o,o}(n)\)), at the same time, denote the number of partitions of \(n\) into odd number of distinct parts in which the smallest part is even (respectively, odd).

Then by interpreting (2.6) and (2.7) we get the partition theory interpretations of mock theta function \(\phi(-q)\) and \(\psi(-q)\):

**Theorem 2.3**

\[
(-q; q)_{\infty} (1 - \phi(-q)) = \sum_{m=1}^{\infty} (D_{o,e}(m) + D_{o,o}(m) + D_{e,e}(m) - D_{e,o}(m)) q^m \tag{2.8}
\]

\[
(-q; q)_{\infty} (-\psi(-q)) = \sum_{m=1}^{\infty} D_{o,o}(m) q^m \tag{2.9}
\]

**Proof.**

By multiplying (2.9) with 2 minus (2.8) we get that

**Corollary 2.4**

\[
(-q; q)_{\infty} (\phi(-q) - 2\psi(-q) - 1)
\]

\[
= \sum_{m=1}^{\infty} (D_{o,o}(m) + D_{e,o}(m) - D_{e,e}(m) - D_{o,e}(m)) q^m
\]

\[
= \sum_{k \geq 1} (-1)^{k-1} q^k (-q^{k+1})_{\infty}
\]

8
3 Several Applications of The Involution

The involution we use in this section is essentially the same with previous $\alpha$, but strictly, we will denote it as $\alpha'$ to indicate that there are minor differences.

We first prove two theorems from Andrews [6] and Andrews [7]. They serve as responses to Andrews' questions on finding combinatorial interpretations of these problems.

First recall the definition of Frobenius symbol (see Andrews [3]). The Frobenius symbol is two rows of decreasing, non-negative integers of equal length, which is often used as one way of representation of a partition. For example, $\lambda = (7, 7, 6, 4, 4, 2, 2)$ as a partition of 32 have the following Ferrers diagram representation. (Figure 3)

One counts the number of squares in the 4 rows to the right and up of the diagonal, and the number of squares in the 4 columns to the left and under of the diagonal, getting the Frobenius symbol as

$$
\begin{pmatrix}
6 & 5 & 3 & 0 \\
6 & 5 & 2 & 1
\end{pmatrix}
$$

**Theorem 3.1** (Andrews [7], Theorem 4) The number of partitions of $n$ whose Frobenius symbol has no 0 on the top row equals the number of partitions of $n$ in which the smallest number that is not a summand is odd.

In order to prove Theorem 3.1, we first reduce it to proving the following identity:

**Lemma 3.2** (Andrews [7], Lemma 12)

$$
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n^2} = \frac{1}{(q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}
$$
The left hand side is the generating function for partitions of \( n \) whose Frobenius symbol has no 0 on the top row. To see this, we draw a \( n \times (n+1) \) rectangle, attach a partition with the largest part at most \( n \) under the rectangle and then attach a partition with the largest part at most \( n \) to the right of the rectangle.

At the same time, the right hand side is actually generating function of the number of partitions of \( n \) in which the smallest number that is not a summand is odd, since:

\[
\frac{1}{(q)\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} = \frac{1}{(q)\infty} \sum_{n=0}^{\infty} q^{n(2n+1)}(1-q^{2n+1}) = \sum_{n=0}^{\infty} \frac{q^{1+2+\cdots+2n}}{\prod_{j=1}^{\infty} (1-q^j)}
\]

**Proof of Theorem 3.2.** Rewrite lemma 3.2 by multiplying \((q)\infty\) in both sides, we get

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n^2}}{(q)_n} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}
\]

As in step 1 of the proof of Theorem 1.2, every term in the left hand side represents a triple \((\lambda, \mu, \rho_{n+\ell})\) with sign \((-1)^\ell\).

1. \( \lambda \in D \) is a partition with exactly \( n \) distinct parts.
2. \( \mu \in P \) is a partition with at most \( \ell \) parts.
3. \( \rho_{n+\ell} \) is a Sylvester’s triangle.

Similar to the previous \( \alpha \), we define an involution \( \alpha' \) as follows. Compare \( \lambda_1 \) the first part of \( \lambda \) and \( \mu_1 \) the first part of \( \mu \). If \( \lambda_1 \geq \mu_1 \), we remove the first part of \( \lambda \) and attach it to \( \mu \). This move changes the sign since it add 1 to \( \ell \). If \( \lambda_1 < \mu_1 \), we remove the first part of \( \mu \) and attach it to \( \lambda \). This move also changes the sign since it subtract 1 from \( \ell \).

All are canceled except those that with \( \lambda \) empty partition \( \emptyset \), where we have \( n = 0 \). What have left are \((-1)^\ell \rho_\ell\), thus the conclusion.

**Definition.** (Andrews [6]) A partition with initial \( k \)-repetitions is a partition in which if any \( j \) appears at least \( k \) times as a part then each positive integer less than \( j \) appears at least \( k \) times as a part.

In a partition, one part is called a distinct part if it only appears once. Let \( D_e(m, n) \) (resp. \( D_o(m, n) \)) denote the number of partitions of \( n \) with initial 2-repetitions, with \( m \) different parts and an even (resp. odd) number of distinct parts.

**Theorem 3.3** (Andrews [6], Theorem 2)

\[
D_e(m, n) - D_o(m, n) = \begin{cases} (-1)^j, & \text{if } m = j, n = j(j+1)/2 \\ 0, & \text{otherwise} \end{cases}
\]
Proof. We first write down the bivariate generating function as follows:

\[
\sum_{n,m \geq 0} (D_e(m,n) - D_o(m,n)) z^n q^m
\]

\[
= \sum_{n=0}^{\infty} z^n q^{1+2+2+\cdots+n} \frac{(q)_n}{(q)_n} \prod_{j=n+1}^{\infty} (1 - zq^j)
\]

\[
= \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}(q^{n+1})_\infty}{(q)_n}
\]

Then we reduce the proof to the following identity:

\[
\sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}(q^{n+1})_\infty}{(q)_n} = \sum_{j=0}^{\infty} (-1)^j z^j q^{j(j+1)/2}
\]

(3.1)

The above identity can be proved applying the involution \( \alpha' \) while inserting a new variable \( z \) (in order to track the number of parts). By (3.1), let \( z \to q^{-1} \) and \( q \to q^2 \), we have

\[ (q; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q)_2n} = \sum_{j=0}^{\infty} (-1)^j q^j. \]

or, by the classical identity of Euler, \((q; q^2) = 1/(q; q)_\infty\), we have:

\[ \sum_{n=0}^{\infty} \frac{q^{1+2+\cdots+2n}}{(q)_2n} = (-q)_\infty \sum_{j=0}^{\infty} (-1)^j q^j. \]

Let \( q_E(n) \) denote the number of partitions of \( n \) into even number of distinct parts; and \( q(n) \) the number of partitions of \( n \) into distinct parts. So we have proved the following identity combinatorially.

**Theorem 3.4**

\[ q_E(n) = q(n) - q(n-1^2) + q(n-2^2) - q(n-3^2) + \cdots \]

**proof.**

**Remark.** Interestingly enough, for unrestricted partitions, the same conclusion holds. Compare the above proof with the combinatorial proof of the following identity in Yee [18]:

\[ p_E(n) = p(n) - p(n-1^2) + p(n-2^2) - p(n-3^2) + \cdots \]
where $p_E(n)$ and $p(n)$ denote the number of partitions into an even number of parts and the number of partitions, respectively.

Using the same involution of Theorem 3.1, we can also prove the following two identities. We omit the proofs.

**Corollary 3.5**

$$
\sum_{n=1}^{\infty} \frac{q^n}{(q)_{n-1}(q)_n} = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n+1)/2}
$$

or, more generally,

$$
\sum_{n=1}^{\infty} z^n q^{n^2} \frac{(zq^{n+1})}{(q)_{n-1}} = \sum_{n=1}^{\infty} (-1)^{n-1} z^n q^{n(n+1)/2}
$$

**Proof.**

**Corollary 3.6**

$$
\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(zq)_n(q)_n} = \frac{1}{(zq)_\infty}
$$

(3.2)

**Proof.**

**Remark.** Corollary 3.6 can be proved using Durfee square [4]. Here it is one brand new combinatorial proof, though a little more complicated than the standard one.

The next application of the involution is an identity involving another Ramanujans third order mock theta function $f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[1-q^n]^2}$. Fine [15] derived the following identity applying some transformation formulas.

**Theorem 3.7** (Fine [15], pp. 56)

$$
f(q) = 1 + \frac{1}{(-q;q)_\infty} \sum_{k \geq 1} (-1)^{k-1} q^k (-q^{k+1})_\infty
$$

**Remark.** Combinatorially, $f(q) - 1 = \sum_{n=1}^{\infty} (N_e(n) - N_o(n))q^n$, where $N_e(n)$ (respectively, $N_o(n)$) is the number of partitions of $n$ with even (respectively, odd) rank. On the other hand, $\sum_{k \geq 1} (-1)^{k-1} q^k (-q^{k+1})_\infty = \sum_{n=1}^{\infty} (L_e(n) - L_o(n))q^n$, where $L_e(n)$ (respectively, $L_o(n)$) is the number of partitions of $n$ into distinct parts with the smallest part even (respectively, odd). So the above identity relates these two enumeration problems.
Differently with previously involutions, we will compare the smallest parts of $\mu$ and $\lambda$ along the way. All other procedures are similar.

**Proof.** As before, we rewrite this identity as

$$
\sum_{n=1}^{\infty} \frac{q^{n^2}(-q^{n+1})}{(-q)_n} = \sum_{k \geq 1} (-1)^{k-1} q^k (-q^{k+1}) \infty
$$

(3.3)

Every term in the left-hand side represents a triple $(\lambda, \mu, \rho_{n+\ell})$ with sign $(-1)^{k-n+1}$ ($k$ to be determined).

1. $\lambda \in D'$ is a partition with exactly $n$ distinct parts (empty part may be included).
2. $\mu \in P'$ is a partition with $\ell$ parts.(empty parts may be included)
3. $\rho_{n+\ell}$ is a Sylvester’s triangle.

Similar to the previous involution, we define an involution $\alpha''$ as follows. Compare the smallest part of $\lambda$ and $\mu$ small the least part of $\lambda$ and $\mu$ (both could be empty parts). If $\lambda_{\text{small}} \leq \mu_{\text{small}}$, we remove the smallest part of $\lambda$ and attach it to $\mu$. This move changes the sign since it substracts 1 from $n$. If $\lambda_{\text{small}} > \mu_{\text{small}}$, we remove the least part of $\mu$ and attach it to $\lambda$. This move also changes the sign since it a add 1 from $n$.

All are canceled except those $(\lambda, \mu)$ with sign $(-1)^{k-1}$ satisfying that $\lambda$ only contains one part $k - 1$ and $\mu$ contains $\ell$ nonempty parts, each $\geq k - 1$. Then attaching these $k - 1$ under $\mu$, aside with the Sylvester’s triangle $\rho_{k+1}$, we get the right-hand side of the identity.

Combining both the above theorem and Corollary 2.4, we get the following well known relation, first derived by Ramanujan:

**Corollary 3.8**

$$
\phi(-q) - 2\psi(-q) = f(q)
$$

**Proof.**

Just like Corollary 3.5 and Corollary 3.6, we immediately get two identities. We omit the proofs.

**Corollary 3.9**

$$
\sum_{n=1}^{\infty} \frac{q^{n^2 + n}}{(-q)^2_n} = \frac{1}{(-q)_{\infty}} \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} q^{k(k-1)/2} q^{k+n} (-q^{k+n+1}) \infty
$$

(3.4)

$$
\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q)_{n-1}(-q)_n} = \frac{1}{(-q)_{\infty}} \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} q^{k(k+1)/2} q^{k+n+1} (-q^{k+n+2}) \infty
$$

(3.5)
Proof.

As in Theorem 3.3, we can generalize (3.3), (3.4) and (3.5) by inserting a new variable $z$ into the identities to track the number of parts, getting the following:

**Corollary 3.9***

\[ \sum_{n=1}^{\infty} \frac{z^n q^{n^2} (-z q^{n+1})}{(-q)_n} = \sum_{k \geq 1} (-1)^{k-1} zq^k (-z q^{k+1}) \sim \] (3.6)

\[ \sum_{n=1}^{\infty} \frac{z^n q^{n^2+n} (-z q^{n+1})}{(-q)_n} = \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} z^k q^{\frac{k(k-1)}{2}} q^{k+n} (-z q^{n+k+1}) \sim \] (3.7)

\[ \sum_{n=1}^{\infty} \frac{z^n q^{n^2} (-z q^{n+1})}{(-q)_{n-1}} = \sum_{k \geq 1} \sum_{n \geq 1} (-1)^{n-1} z^{k+1} q^{\frac{k(k+1)}{2}} q^{k+n+1} (-z q^{n+k+2}) \sim \] (3.8)

The above three identities can be translated into the language of partitions. We only do this for (3.7), stating it as a theorem, which, as will be seen, is in the spirit of Andrews’ “partition with initial repetitions”. For more details, see Andrews [6].

Let $I_e(m, n)$ (resp. $I_o(m, n)$) denote the number of partitions of $n$ with initial 2-repetitions, with $m$ different parts and an even (resp. odd) number of repeated parts. We call a partition into distinct parts is with a initial Sylvester’s triangle $\rho_k$ when it contains 1, 2, \ldots, $k$ and does not contain $k+1$ and it is not $\rho_k$ itself. Let $S_e(m, n)$ (resp. $S_o(m, n)$) denote the number of partitions of $n$ with initial sylvester’s triangle, with $m$ different parts, and the first gap (i.e. the difference between neighboring parts which is larger than one) of parts is even (resp. odd).

**Theorem 3.10**

\[ I_e(m, n) - I_o(m, n) = S_o(m, n) - S_e(m, n) \]

Proof. Since in the right-hand side of (3.7) the coefficient of $z^n q^n$ is $S_o(m, n) - S_e(m, n)$. To finish the proof, we only need to rewrite the left-hand side of (2.8) as follows:

\[ \sum_{n=1}^{\infty} \frac{z^n q^{n^2+n} (-z q^{n+1})}{(-q)_n} \]

\[ = \sum_{n=0}^{\infty} \frac{z^n q^{1+2+2+\ldots+n\cdot2}}{(-q)_n} \prod_{j=n+1}^{\infty} (1 + zq^j) \]

\[ = \sum_{n,m \geq 0} (I_e(m, n) - I_o(m, n)) z^m q^n \]
4 Flushed Partitions, Concave Compositions and Proper Partitions

We first prove two identities as follows:

**Lemma 4.1**

\[ \sum_{n=1}^{\infty} q^{n(3n-1)/2} (1 - q^n) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1 + q) \cdots (1 + q^n)}. \quad (4.1) \]

\[ \sum_{n=1}^{\infty} q^{n(3n-1)/2} (1 - q^n) = \sum_{n=1}^{\infty} \frac{q^{n(2n-1)}}{(-q)2n}. \quad (4.2) \]

**Proof.** For the second equality, let \( P_o(D_n) \) (resp. \( P_e(D_n) \)) denote the number of partitions \( \lambda \) of \( n \) with distinct parts and the largest part is odd (resp. even). The following identity is due to Fine [14] which can be reached by the well-known Franklin’s involution.

\[ \sum_{n=1}^{\infty} q^{n(3n-1)/2} (1 - q^n) = \sum_{n=1}^{\infty} (P_o(D_n) - P_e(D_n)) q^n. \quad (4.3) \]

Now it is sufficient to prove the following

\[ \sum_{n=1}^{\infty} (P_o(D_n) - P_e(D_n)) q^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1 + q) \cdots (1 + q^n)}. \quad (4.4) \]

Identity (4.4) can be shown as follows. Given a partition with distinct \( n \) parts \( \lambda \in D_n \), we split the partition into a Sylvester’s triangle \( \rho_n = (n, n-1, \cdots, 1) \) and a partition \( \mu \in P \). The conjugate of \( \mu \) is a partition, \( \mu^* \), with the largest part at most \( n \) and with the number of parts \( r \). Note that \( \lambda_1 \), the largest part of \( \lambda \) is \( n + r \). So in the left hand side of (4.4), the above \( \lambda \) is actually assigned with \( (-1)^{n-1+r} \).

The right hand side of (4.4), at the same time, is the generating function of a pair of partitions \((\rho_n, \mu^*)\), where \( \rho_n \) is Sylvester’s triangle and \( \mu^* \) has largest part no more than \( n \) and with \( r \) parts. Each term is assigned with \( (-1)^{n-1+r} \), thus the conclusion.

The proof of the second identity is in the same fashion of (4.4), while the parity of the Sylvester’s triangle should be noted. Again, for a partition of \( n \) with distinct parts \( \lambda \in D_n \) signed with \( (-1)^{\lambda_1} \), we split it into a Sylvester’s triangle \( \rho_{2n-1} = (2n-1, 2n-2, \cdots, 1) \) and a partition \( \mu \) with at most \( 2n \) parts, signed with \( (-1)^{\mu_1} \), where \( \mu_1 \) is the largest part of \( \mu \).

**Definition:** Durfee Symbols was introduced by Andrews in [5], which is defined as follows: Using Corollory 2.6, we can represent a unrestriced partition as its \( n \times n \) Durfee square.
and two partitions with the largest part at most \( n \). We then denote the partition with two rows of integers, the top row listing the parts of the conjugate of the partition to the right of the Durfee square and the bottom row listing the parts of the partition under the Durfee square. We also write down a subscript \( n \) in the end to denote that the Durfee square is \( n \times n \). Take the partition \( \lambda = (11, 11, 11, 9, 7, 5, 5, 4, 4, 3) \) for example, whose Ferrers diagram is depicted in Figure 4, The Durfee symbol representation of \( \lambda \) is as follows:

\[
\begin{pmatrix}
5 & 5 & 4 & 4 & 3 & 3 \\
5 & 5 & 4 & 4 & 3 & 0
\end{pmatrix}_5
\]

Suppose the Durfee square of a partition is \( n \times n \), we then call a partition proper partition when its Durfee symbol has the same number of \( n \)'s in both the top and bottom rows. All other partitions are called improper partitions. The number of proper partitions of \( n \) is denoted as \( PR(n) \). The number of improper partitions of \( n \) is denoted as \( IMPR(n) \). A typical example of proper partition \( \lambda = (11, 11, 11, 9, 7, 5, 5, 4, 4, 3) \) has its Ferrers diagram in Figure 4.

![Figure 4: \( \lambda = (11, 11, 11, 9, 7, 5, 5, 4, 4, 3) \)](image)

Obviously, the generating function of proper partitions is

\[
\sum_{n=0}^{\infty} PR(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q)_n^2(1 - q^{2n})}
\]

As stated in the introduction, a partition \( \lambda \) is called \textit{flushed} if the smallest part to appear an even number of times is even. The unflushed partition are, of course, those partitions in which the smallest part to appear an even number of times is odd. We denote the number of flushed partitions of \( n \) as \( F(n) \).

In a flushed partition, suppose 1, 2, \ldots, 2i - 1 all appear an odd number of times, and 2i appear an even number of times (zero times included). We extract 1 + 2 + \cdots + 2i - 1 and left 1, 2, \ldots, 2i all appearing an even number of times. So one easily writes down the generating function of \( F(n) \) as follows:
\[
\sum_{n \geq 0} F(n) q^n = \sum_{n=1}^{\infty} \frac{q^{n(2n-1)}}{(q^2; q^2)_{2n}} = \sum_{n=1}^{\infty} \frac{q^{n(3n-1)/2}(1 - q^n)}{(q)_\infty} \quad (4.5)
\]

where the second equality is by (4.2).

Relating Lemma 4.1 and Theorem 2.1 we get the following:

**Theorem 4.2 (Generating Functions for Flushed Partitions)** The number of flushed partitions of \( n \) is equal to the number of proper partitions of \( n \). So, they share the same generating function,

\[
\sum_{n \geq 0} F(n) q^n = \sum_{n=0}^{\infty} PR(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n).
\]

**Proof.**

**Remark:** In [13] Dyson defined the rank of a partition as the largest part minus the number of parts. We denote the number of partitions of \( n \) with rank \( m \) by \( N(m, n) \). Then in [10] Atkin and Swinnerton-Dyer derived the generating function of \( N(0, n) \) as follows:

\[
\sum_{n=0}^{\infty} N(0, n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2}(1 - q^n)
\]

It would be interesting to compare this with Theorem 4.2. Notice that there is some kind of symmetry between proper partitions and unrestricted partitions with rank 0. Since by the classical the pentagonal number theorem, we have \((q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^n(3n-1)/2 (1 + q^n)\), this identity and theorem 4.2 reveal the relations of pentagonal number theorem with three different variations of signs.

**Theorem 4.3 (Generating Functions for Concave Compositions of Even Length)**

The number of concave compositions of even length of \( n \) is equal to the number of improper partitions of \( n \) and thus is equal to the number of unflushed partitions of \( n \). So, they share the same generating function,

\[
\sum_{n=0}^{\infty} cc(n) q^n = \sum_{n=0}^{\infty} IMPR(n) q^n = \frac{1}{(q)_\infty} \left(1 - \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n)\right).
\]

**Remark:** Theorem 1.1 is the combination of theorem 4.2 and theorem 4.3.
Proof. We will construct a bijection to finish the proof. Define a map $\phi$ from concave compositions of even length of $n$ to unstricted partitions of $n$ as follows. First take a concave composition $C$:

$$a_1 > a_2 > \cdots > a_m = b_m < b_{m-1} < \cdots < b_1,$$

the partition $\phi(C)$ will depend on four different situations.

1. When $a_m = b_m = 0, a_{m-1} = b_{m-1}$, let the first row of the Ferrers diagram have $a_1$ squares. Then draw $b_1$ squares under the square $(1, 1)$; Begin with the position $(2, 2)$, $a_2$ squares are put in the second row. Then we put $b_2$ squares under the position $(2, 2)$. Continue in this fashion, we get the tabeaux. For example, if the concave composition is $C : 2 > 1 > 0 = 0 < 1 < 2$, we have $\phi(C) = (2, 2, 2)$ as illustrated in Figure 5. In this situation, the resulting partition is an improper partition, about which the bottom row of the Durfee symbol have one more $n$ than the top row of the durfee symbol, when the Durfee square is $n \times n$.

\[
\begin{array}{|c|c|}
\hline
\end{array}
\]

Figure 5: concave compositions with two least equal parts

2. When $a_m = b_m \neq 0$, we do symmetrically as in situation number one. let the first column of the Ferrers diagram have $b_1$ squares. Then draw $a_1$ squares to the right of the square $(1, 1)$; Begin with the position $(2, 2)$, $b_2$ squares are put in the second column. Then we put $a_2$ squares to the right of the position $(2, 2)$. Continue in this fashion, we get the Ferrers diagram. For example, if the concave composition is $C : 2 > 1 = 1 < 2$, we have $\phi(C) = (2, 2, 2)$ as illustrated in Figure 6. In this situation, the resulting partition is an imroper partition, about which the bottom row of the Durfee symbol have one less $n$ than the top row of the durfee symbol, when the Durfee square is $n \times n$.

\[
\begin{array}{|c|c|}
\hline
\end{array}
\]

Figure 6: concave compositions without zeros
3. When $a_m = b_m = 0, a_{m-1} \neq b_{m-1}$, if $a_{m-1} > b_{m-1}$, we follow the procedure of situation number one, only to find the resulting partition a proper partition. We then make one more move: take one largest part to the right of the Durfee square and put it under the Durfee square. For example, if the concave composition is $C : 3 > 2 > 0 = 0 < 1 < 2$, we have $\phi(C) = (4, 4)$ as illustrated in Figure 7. In this situation, the resulting partition is an improper partition, about which the bottom row of the Durfee symbol have at least two less $n$'s than the top row of the durfee symbol, when the Durfee square is $n \times n$.

\[ 3 > 2 > 0 = 0 < 1 < 2 \rightarrow \begin{array}{|c|c|c|} \hline \ 0 \ & \ 1 \ & \ 2 \ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline \ 0 \ & \ 1 \ & \ 2 \ \hline \end{array} \]

Figure 7: the map $\phi$: $3 > 2 > 0 = 0 < 1 < 2 \rightarrow (4, 4)$

4. When $a_m = b_m = 0, a_{m-1} \neq b_{m-1}$, if $a_{m-1} > b_{m-1}$. We follow the procedure of situation number one. In this situation, the resulting partition is an improper partition, about which the bottom row of the Durfee symbol have at least two more $n$’s than the top row of the durfee symbol, when the Durfee square is $n \times n$.

One easily observes that $\phi$ is a bijection, which map all concave compositions of even length of $n$ into improper partitions of $n$.

Remark. In step 3 of the above proof, the last move we made is the removal of certain part under the Durfee square and the replacement of this part to the right of the Durfee square, this technique was also used by Alladi to represent the number of unrestricted partition as weighted sum of the number of partitions of $n$ with differences $\geq 2$ between parts. Such move was defined by Alladi as sliding operation on Ferrers graphs. For more details, see Alladi [1].

In order to prove Sylvester’s first problem, we have the following Corollary of the above theorems.

**Corollary 4.4** The number of unflushed partitions of $n$ with $m$ parts is equal with the number of unstricted partitions of $n$ with $m$ parts in which when the Durfee square is $k \times k$, the Durfee symbol cannot contain even number of $k$’s in the bottom row and no $k$’s in the top row.

**Proof.** Just like what we have done in the previous two sections, we try to insert another variable $z$ into the identities and start all the procedures of proving generating function of unflushed partitions all over again. Then the generating function for unflushed partitions of $n$ with $m$ parts can be denoted as:
the first identity is by an analysis similar to that of (4.5), the second identity is from the proof of Lemma 4.1, and the third identity is by (2.3) and (3.2).

We interpret the first and the last term of the above identity and get the conclusion.

We conclude this paper with the proof of Sylvester’s problem.

**Theorem 4.5** Unflushed partitions of $n$ with odd number of parts are equinumerous with unflushed partitions of $n$ with even number of parts.

**Proof.** By Corollary 4.4, we only need to analyze the parity of the number of unstricted partitions of $n$ with $m$ parts, in which, when the Durfee square is $k \times k$, the Durfee symbol cannot contain even number of $k$’s in the bottom row and no $k$’s in the top row.

If the total number of $k$’s in the Durfee symbol is odd, say, $2i - 1$, then we can arrange these $k$’s in top and bottom rows in $2i$ ways, by putting $0, 1, \cdots, 2i - 1$ copies of $k$’s in the bottom row and the rest $k$’s in the top row. Observe that the parity of number of parts changes accordingly, that is, among these $2i$ partitions, $i$ ones have even number of parts and $i$ ones have odd number of parts.

If, however, the total number of $k$’s in the Durfee symbol is even, say, $2i$, then we can also arrange these $k$’s in top and bottom rows in $2i$ ways, by putting $0, 1, \cdots, 2i - 1$ copies of $k$’s in the bottom row and the rest in the top row. In this situation the parity of number of parts changes accordingly, too, that is, among these $2i$ partitions, $i$ ones have even number of parts and $i$ ones have odd number of parts.

**References**


